Math 6010 - Assignment 5

(5) Show that Σ^0_n for $n \in \mathbb{N}$ is closed under bounded quantifiers.

Proof. Let $P(x,t)$ be Σ^0_n. We have to show that

$$E(x,z) \equiv \exists t < z \; P(x,t)$$
$$U(x,z) \equiv \forall t < z \; P(x,t)$$

are Σ^0_n. Note that the bound z is an argument of E, U, respectively. Hence neither E, U cannot be reduced to a fixed number of disjunctions, conjunctions (that is, to exercise 4).

Instead we use induction on n:

Base case: Assume $P(x,t)$ is Σ^0_0, that is, its characteristic function c_P is computable.

Then the characteristic function of U is

$$c_U(x,z) := \prod_{t=0}^{z-1} c_P(x,t),$$

hence also computable. Thus U is Σ^0_0.

Similarly the characteristic function

$$c_E(x,z) := 1 - \prod_{t=0}^{z-1} (1 - c_P(x,t))$$

of E is computable. Hence E is Σ^0_0.

Induction step: Let $n \geq 1$ and $P(x,t) \equiv \exists y \; R(x,y,t)$ for some R in Π^0_{n-1}.

Clearly E is Σ^0_n since Σ^0_n is closed under arbitrary existential quantifiers as proved in class.

For U we claim that

$$(1) \qquad \forall t < z \, \exists y \; R(x,y,t) \;\equiv\; \exists y \, \forall t < z \; R(x,(y)_t,t).$$

Assume $U(x,z)$ (i.e. the left hand side of (1)) holds. For $t < z$ let y_t be a witness such that $R(x,y_t,t)$. Define $y := \prod_{t=0}^{z-1} p_t^{y_t}$. Then $\forall t < z \; R(x,(y)_t,t)$. Hence the right hand side of (1) holds.

The converse implication in (1) is clear. Hence (1) is proved.

Since Π^0_{n-1} is closed under universal quantifiers and substitution with total computable functions, $\forall t < z \; R(x,(y)_t,t)$ is Π^0_{n-1}. Thus (1) yields that $U(x,z)$ is Σ^0_n. □