Completeness

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Reductions

Definition

A many-one reduction f from A to B is **first order** (f.o.) if the graph of f is first order definable.

Since FO $\subsetneq$ L $\subseteq$ P, first order reductions are weaker than logspace reductions which are weaker than polytime reductions.

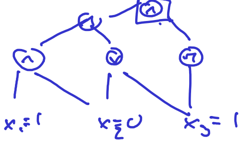
Complete Problems in L and NL

- in L (via f.o. reductions):
  Cycle: Given an undirected graph, does it contain any cycle?
- in NL (via f.o. reductions):
  2SAT
  Reachability: Given a directed graph and vertices s, t,
  is there a path from s to t?

Proof idea

- Let C be a class of structures that is decided by a nondeterministic TM N in logspace.
- Show that for any $\mathbf{A}$ the configuration graph $G(\mathbf{A})$ of N on input $\mathbf{A}$ is f.o.-definable.
- Then $\mathbf{A} \in C$ iff in $G(\mathbf{A})$ there is a path from start to accept configuration.

Complete in P



- via f.o. reductions:
  Circuit-Value-Problem: Given a Boolean circuit with specified input, is the output 1?
- via NC reductions:
  Given a DTM M and n in unary, does M halt on an empty tape in $\leq n$ steps?
  - NC := problems for which there exist c, k such that they are decidable in time $O((\log n)^c)$ using $O(n^k)$ parallel processors (Nick's class/problems that can be efficiently parallelized)
  - Typical problems in NC:
    matrix multiplication, determinant, polynomial gcd
  - $\text{NL} \subseteq \text{NC} \subseteq \text{P}$
- via logspace reductions:
  Linear Programming: Maximize a linear function subject to linear inequality constraints.

Complete in NP

SAT via f.o. reductions.

Complete in PSPACE (under f.o. reductions)

QBF: Given a quantified Boolean formula,

$$\exists x_1 \; \forall x_2 \; \ldots Q x_n : \; \alpha(x_1, \ldots, x_n)$$

is it satisfiable?

Proof idea for PSPACE-hardness with polytime reductions

- Let L be a language that is decided by a TM M in space $s(n)$, let $x \in \{0,1\}^n$.
- Construct a QBF ψ of size $O(s(n)^2)$ that is true iff $x \in L$.
- For configurations $d, e \in 2^{cs(n)}$ of M, define
  $\psi_0(d,e) := d = e$ or e is a successor of d;
  $\psi_i(d,e) := \exists f \; \psi_{i-1}(d,f) \wedge \psi_{i-1}(f,e)$ for $i > 0$.
- Then $x \in L$ iff $\psi_{cs(n)}(\text{start,accept})$ holds.
  But $\psi_{cs(n)}$ has length exponential in $s(n)$.
- To get polynomial length, note $\psi_i(d,e) \equiv$
  $\exists f \forall v \forall w \; [(v = d \wedge w = f) \vee (v = f \wedge w = e)] \Rightarrow \psi_{i-1}(v,w)$

Complete in EXPTIME

Given a DTM M and n in binary,
does M halt on an empty tape in $\leq n$ steps?