Descriptive Complexity

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Idea

Measure the complexity of a class of structures (e.g. graphs) by the logical formula that describes it.

Example

A graph (V, E) has a 3-cycle iff

satisfies

$$(V, E) \models \exists x \; \exists y \; \exists z : \; x \neq y \neq z \neq x \wedge E(x, y) \wedge E(y, z) \wedge E(z, x).$$

Described by first order (f.o.) sentence, hence in FO $\subseteq$ L.

Definition (Structures)

A **relational signature** ρ is a sequence of relational symbols $R_1, \ldots, R_k$ each with an arity $n_1, \ldots, n_k > 0$ and constant symbols $c_1, \ldots, c_\ell$.

$\mathbf{A} := (A, R_1^{\mathbf{A}}, \ldots, R_k^{\mathbf{A}}, c_1^{\mathbf{A}}, \ldots, c_\ell^{\mathbf{A}})$ is a **relational structure** over ρ if each $R_i \in \rho$ has an interpretation $R_i^{\mathbf{A}} \subseteq A^{n_i}$ as n_i-ary relation on A and each $c_j \in \rho$ has an interpretation $c_i^{\mathbf{A}} \in A$ as constant in A.

Definition

Let FO be the set of first order sentences, i.e., sentences of the form

$$\exists x_1 \, \forall x_2 \, \ldots Q x_n : \; \alpha(x_1, \ldots, x_n)$$

where α is a quantifier-free sentence in some relational signature ρ (and $=$).

FO $\subseteq$ L

Theorem

If a class of finite ρ-structures is definable in FO, it is decidable in L.

Proof.

Let

$$\varphi := \exists x_1 \, \forall x_2 \; \dots Q x_n : \; \alpha(x_1, \dots, x_n)$$

be a f.o. sentence over ρ.
Construct a logspace DTM M such that for all ρ-structures $\mathbf{A}$,

$$\mathbf{A} \models \varphi \text{ iff } M \text{ accepts } \mathbf{A}.$$

Base case $n = 0$: Then $\varphi = \alpha$ is a fixed variable-free Boolean combination of atomic formulas

$$R_i(c_1, \dots, c_{n_i}), \quad c_u = c_v$$

for constants $c_1, \dots, c_{n_i}, c_u, c_v$.

- Whether $\mathbf{A} \models \varphi$ can be checked in $O(\log |\mathbf{A}|)$ space.

Ex. $\mathcal{S} = (E, \underbrace{c_1, c_2}_{\text{constants}})$ — E binary

$\alpha = (E(c_1, c_2) \vee E(c_2, c_1)) \wedge c_1 \neq c_2$

$\underline{\mathcal{A}} = (\{1, \dots, n\}, E^{\mathcal{A}}, c_1^{\mathcal{A}}, c_2^{\mathcal{A}})$ — graph with 2 special vertices

To check $\mathcal{A} \models \alpha$

- record $c_1^{\mathcal{A}}, c_2^{\mathcal{A}} \in \{1, \dots, n\}$ — $O(\log n)$ space
- lookup $E^{\mathcal{A}}(c_1^{\mathcal{A}}, c_2^{\mathcal{A}}) = 0/1$ in $n \times n$ table in input — constant space
- evaluate $E^{\mathcal{A}}(c_1^{\mathcal{A}}, c_2^{\mathcal{A}}) \vee E^{\mathcal{A}}(c_2^{\mathcal{A}}, c_1^{\mathcal{A}})$ — – " –

Let

$$\psi(x_1) := \forall x_2 \ \ldots Qx_n : \ \alpha(x_1, \ldots, x_n).$$

- Let M_0 be a DTM that computes $\psi(c)$ for $c \in A$ in logspace.
- To check $\varphi := \exists x_1 \psi(x_1)$, test all assigments of x_1 in A with M_0.
- **Correctness:** $\mathbf{A} \models \varphi$ iff M_0 yields $\psi(c) = 1$ for some $c \in A$.
- **Complexity:** $O(\log |A|)$ space for recording the current value of x_1 and $O(\log |\mathbf{A}|)$ space for M_0.
- Similar for the case $\varphi := \forall x_1 \ \psi(x_1)$.
- Hence $\mathbf{A} \models \varphi$ is decidable in L. □

Note

FO $\subsetneq$ L

[Parity can be computed in logspace but not expressed in FO.]

"Parity: Input $w \in \{0,1\}^*$

Q: What is the number of 1's in w mod 2 ?

Second order logic

Example

(V, E) is 3-colorable iff

$$(V, E) \models \exists R\ \exists G\ \exists B\ \forall x\ \forall y :\ [R(x) \vee G(x) \vee B(x)]$$
$$\wedge [E(x, y) \Rightarrow ((R(x) \wedge R(y))' \wedge (G(x) \wedge G(y))' \wedge (B(x) \wedge B(y))')]$$

- Colors R, G, B are unary predicates (sets of vertices).
- Because of the quantification over sets, this is a second order sentence, hence in SO$\exists$.

Definition (Existential second order logic)

Let SO$\exists$ be the set of existential second order sentences, i.e., sentences of the form

$$\exists R_1 \ldots \exists R_m\ \varphi$$

for relational symbols R_i and a first order sentence φ.

SO∃ = NP

Fagin's Theorem (1973)

A class of finite structures is definable in existential second order logic iff it is decidable in NP.

Proof idea.

$\Rightarrow$ Given a structure **A** and a sentence $\exists R_1 \ldots \exists R_m\ \varphi$, a nondeterministic TM can guess an interpretation of $R_1, \ldots, R_m$ in polynomial time and then verify the f.o. φ in logspace.

$\Leftarrow$ Let N be a nondeterministic TM that runs in polynomial time. Write a sentence

$$\Phi = \exists C_1 \ldots \exists C_g \exists D : \ \varphi$$

saying:

"There exists a path of consecutive configurations from start to an accepting configuration."

Assume N runs in time $\leq n^k$ on inputs of length n.
Wlog $|\Delta(q, a)| = 2$ for all $q \in Q, a \in \Gamma$.

- View the configurations of a computational path of N as rows of an $n^k \times n^k$ table T with entries in $\Gamma \cup (\Gamma \times Q) =: \{\gamma_1, \ldots, \gamma_g\}$.
- C_i is a $2k$-ary relation such that $C_i(t, s) = 1$ if the entry of T at time $t \in [n]^k$, space $s \in [n]^k$ is γ_i; else 0.
- D is a k-ary relation such that $D(t) = 1$ if N takes transition 1 at step t; $D(t) = 0$ if N takes transition 0.
- φ is the Boolean formula stating that $C_1, \ldots, C_g, D$ encode a valid accepting computation of N as in the proof that SAT is NP-complete.

Then N accepts **A** iff **A** satisfies Φ. $\square$

Inductive definitions

Transitive closure of edge relation E for graphs is not f.o. definable:

$$E^*(x, y) := x = y \vee \exists z[E(x, z) \wedge E^*(z, y)]$$

This inductive definition can be formalized via a least fixed point (LFP) operator.

Definition (FO(LFP))

Let FO(LFP) be the language of f.o. inductive definitions (FO with a least fixed point operator).

Theorem (Vardi 1982, Immerman 1982)

P = FO(LFP)

Reductions

Definition

A many-one reduction f from A to B is **first order** (f.o.) if the graph of f is first order definable.

Since FO $\subseteq$ L $\subseteq$ P, first order reductions are weaker than logspace reductions which are weaker than polytime reductions.

Complete Problems

- in L (via f.o. reductions):
  Cycle: Given an undirected graph, does it contain any cycle?
- in NL (via f.o. reductions):
  Reachability: Given an undirected graph an vertices s, t, is there a path from s to t?
- in P (via NC reductions):
  Given a DTM M and n in unary, does M halt on an empty tape in $\leq n$ steps?
- in NP (via f.o. reductions):
  SAT
- in PSPACE:
  QBF
- in EXPTIME:
  Given a DTM M and n in binary, does M halt on an empty tape in $\leq n$ steps?