More NP-complete problems

Peter Mayr

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Cliques

- $K_n = (\{1, \ldots, n\}, \neq)$ … **complete (undirected) graph** on n-vertices
- A graph G has an n-**clique** if K_n embeds into G.
- CLIQUE $:= \{(G, n) \ : \ G$ is a graph with n-clique$\}$

Theorem
CLIQUE is NP-complete.

Proof.
CLIQUE $\in$ NP since a guessed n-clique can be verified in polynomial time.

Claim: 3SAT $\leq_m^p$ CLIQUE

Given a 3SAT instance

$$\Phi = (a_1 \vee b_1 \vee c_1) \wedge \cdots \wedge (a_n \vee b_n \vee c_n)$$

with literals a_i, b_i, c_i.

For the reduction construct a graph G with

- $3n$ vertices labelled $a_1, b_1, c_1, \ldots, a_n, b_n, c_n$
- edges between any 2 vertices except within any triple a_i, b_i, c_i representing a clause of Φ and between a variable and its negation.

Φ with $3n$ literals yields a graph with $3n$ vertices in polytime.

Example: $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_1' \vee x_2' \vee x_3') \wedge (x_1' \vee x_2 \vee x_3)$

Claim: Φ is satisfiable iff G has an n-clique

$\Rightarrow$:

- Assume Φ has a satisfying assignment.
- In each triple a_i, b_i, c_i of G choose a vertex corresponding to a true literal in this satisfying assignment.
- These n vertices are each pairwise connected, hence a clique.

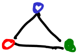
$\Leftarrow$:

- Assume G has an n-clique.
- Then any 2 vertices in that clique are in distinct clauses.
- Assign truth values to variables in Φ such that each literal labelling a vertex in the clique is true (possible since x_j, x_j' are not connected).
- Since each clause contains a vertex from the clique, this assignment satisfies Φ.

$\square$

Graph coloring

- A graph G is n-**colorable** if its vertices can be colored in n colors such that any adjacent vertices have distinct colors.
- Equivalently, G is n-colorable iff there exists a homomorphisms $G \to K_n$.

  $G \to$ 

- n-Coloring $:= \{G \;:\; G \text{ is } n\text{-colorable}\}$.

Theorem

3-Coloring is NP-complete.

Proof.

3-Coloring $\in$ NP since a guessed coloring can be verified in polynomial time.

Claim: 3SAT $\leq_m^p$ 3-Coloring

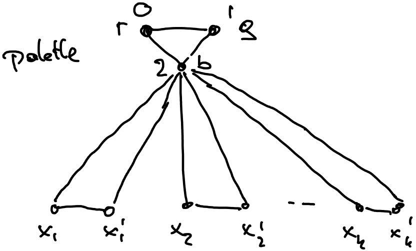
Given a 3SAT instance

$$\Phi = (a_1 \vee b_1 \vee c_1) \wedge \cdots \wedge (a_n \vee b_n \vee c_n)$$

with literals a_i, b_i, c_i.

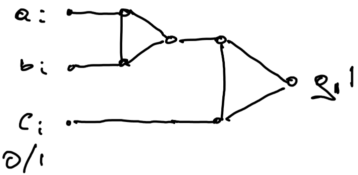
Construct G that is 3-colorable iff Φ is satisfiable as follows:

- Truth assignments of $x_1, \ldots, x_k$ correspond to colors $0, 1$ of vertices.



x_i, x_i' have complementary colors $0, 1$

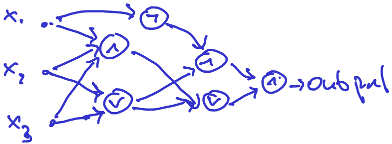
- For each clause $a_i \vee b_i \vee c_i$ connect the vertices corresponding to a_i, b_i, c_i by a **gadget graph** implementing "or".



3-colorable iff one of a_i, b_i, c_i is colored 1 (HW)

Further NP-complete problems (Karp, 1972)

- **kSAT**
- **Circuit Satisfiability Problem**
  Given a Boolean circuit (in gates $\wedge, \vee, \neg$), is there an assignment of inputs $x_1, \ldots, x_n$ that yields output 1?



input more compact than Boolean formula for SAT

- **CLIQUE**
- **Graph** k**-Coloring** for $k \geq 3$
- **Graph Homomorphism Problem**
  Given graphs G, H, is there a homomorphism $G \to H$?

- **Hamiltonian Cycle**
  Given a (di)graph G, is there a ~~path~~ cycle that visits every vertex exactly once?
- **Travelling Salesman Problem** (decision version)
  Given a
  graph G with edges of specified integer weights and $\ell \geq 0$, does G have a Hamiltonian cycle with edges whose weight sum is $\leq \ell$?
- **Exact Cover**
  Given subsets $A_1, \ldots, A_k \subseteq \{1, \ldots, n\}$, is $\{1, \ldots, n\}$ the disjoint union of some A_i?
- **Knapsack (Subset Sum)**
  Given integers $a_1, \ldots, a_n$ and s, does a non-empty subset of the a_i sum to s?
- **MaxCut** (decision version)
  Given a graph G and $k \in \mathbb{N}$, is there a cut of size at least k in G (a partition of vertices into 2 sets A, B with $\geq k$ edges between A and B)?
- **Sudoku** for n^2 numbers