The Immerman-Szelepcsényi Theorem

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Note

- ▶ The complement of a complexity class C is denoted co-C.
- ▶ Clearly deterministic complexity classes are closed under complements but is not known whether NP = co-NP.
- ▶ The following was proved independently by Immerman and Szelepcsényi (then an undergrad) in 1987 and won them the Gödel prize in 1995.

Immerman-Szelepcsényi Theorem

$NSPACE(s(n)) = co\text{-}NSPACE(s(n))$ for $s(n) \geq \log n$.

Corollary

NL=co-NL

Proof.
Idea: Suppose we have a non-deterministic membership test for some $A \subseteq \Sigma^n$ and we know $|A|$.

Then we also have a non-deterministic membership test for $\Sigma^n \setminus A$:

- $x \in \Sigma^n \setminus A$ iff we can successively guess $|A|$ distinct strings $\neq x$ and show they are in A.

Let M be a nondeterministic TM with input and work tape that runs in $s(n)$-space.

▶ For fixed input of length n, every configuration of M (state, positions of tape heads, content of work tape) can be represented by an $s(n)$-tuple over some finite $C = \Gamma \cup \Gamma \times (Q \cup \{0,1\})$ (uses $s(n) \geq \log n$ to encode the position $\leq n$ on input tape).

▶ Let start, accept $\in C^{s(n)}$ be the unique starting and accepting configuration on inputs of length n.

▶ For $m \in \mathbb{N}$ let A_m be the set of configurations that can be reached from start in $\leq m$ steps.

▶ Then M accepts x iff accept $\in A_{|C|^{s(n)}}$.

A non-deterministic TM N with $L(N) = \overline{L(M)}$

On input x of length n, compute $|A_0|, |A_1|, \ldots, |A_{|C|^{s(n)}}|$ inductively:

1. $|A_0| := 1$

2. Given $|A_m|$, set $|A_{m+1}| := 0$ and enumerate $\beta \in C^{s(n)}$ lexicographically.

3. To check $\beta \in A_{m+1}$,
   - 3.1 non-deterministically guess the $|A_m|$ elements $\alpha \in A_m$ in lexicographical order,
   - 3.2 guess and verify a path of length $\leq m$ from `start` to α and
   - 3.3 check that β is a successor of α.

4. If $\beta \in A_{m+1}$, then increase the counter for $|A_{m+1}|$ by 1. Repeat for the next β in lexicographical order.

For $m = |C|^{s(n)}$ enumerate $\alpha \in A_{|C|^{s(n)}}$ in lexicographical order as in steps 3.1-2 above.

If $\texttt{accept} \neq \alpha$ for all of them, then $x \notin L(M)$ and N accepts x.

Space complexity of N:

- At each step N's working tape holds m, $|A_m|, |A_{m+1}|$, β, α, i and 2 intermediate configurations in A_i, A_{i+1} on a path from `start` to α.
- Each of these takes $s(n)$ space.

Hence N runs in $O(s(n))$ space.

Note
To avoid the assumption that $s(n)$ is space constructible, let N run for successive values $s = \log n, \log n + 1, \ldots$

- Whenever M tries to use more than s space, restart N with $s + 1$ space.
- Since M runs in $s(n)$ space, it eventually will not reach any longer configurations and the loop above stops with N only ever using $O(s(n))$ space. $\square$