PSPACE and Savitch's Theorem

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Recall: We had a polytime algorithm for the following.

Reachability (Path)

Input: digraph $G = (V, E)$ with vertices $\{1, \ldots, n\}$
Question: Is there a path in G from 1 to n?

Theorem (Savitch 1970)

Reachability is in DSPACE($(\log n)^2$).

Proof.

Let $G = (V, E)$ be a digraph with $V = \{1, \ldots, n\}$.

- Define Path$(x, y, i) :=$ $\exists$ path in G from i to j of length $\leq 2^i$
- y is reachable from x iff Path$(x, y, \lceil \log_2 n \rceil)$.

Recursion for Path(x, y, i):

1. If $i = 0$, then return $[x = y$ or $(x, y) \in E]$.
2. For $z \in V$ do
3. If Path$(x, z, i-1)$ and Path$(z, y, i-1)$, then return `true`.
4. Return `false`.

Correctness:

- If there is a path from x to y of length $\leq 2^i$, then a midway point z will be found in the loop 2-3 and `true` returned in 3.
- If there is no path from x to y of length $\leq 2^i$, the condition in 3. is never satisfied. Hence `false` is returned in 4.

Input size of Path(x, y, i): $O(\log(n))$ since $x, y, i \leq n$.

Space:

- Path(x, y, i) has recursion depth i. Its computation is represented by a binary tree with 2^i leaves at $i = 0$.
- At each level we need to store x, y, z, i, the value of Path$(x, z, i-1)$, etc. This needs $O(\log n)$ space.
- Hence at any time Path(x, y, i) needs $O(i \log n)$ space.
- Path$(1, n, \lceil \log_2 n \rceil)$ needs $O((\log n)^2)$ space. □

Savitch's Theorem

$\mathrm{NSPACE}(s(n)) \subseteq \mathrm{DSPACE}(s(n)^2)$ for any $s(n) \geq \log n$.

Proof.

Let N be a non-deterministic TM with input (no output) and one working tape that runs in space $s(n)$.

- Consider the configurations of N as vertices of a digraph G with an edge $u \to v$ if v is a successor configuration of u.
- N accepts input x iff there is a path from the starting configuration `start` to some accepting configuration `accept` in G (wlog `accept` is unique).
- Recall there exists c such that for $|x| = n$, N has $\leq 2^{cs(n)}$ reachable configurations (vertices of G).
- Use the recursive algorithm from the previous Theorem to decide Path(`start`, `accept`, $cs(n)$).
  - We never need to store all vertices of G at once, just $O(s(n)^2)$.
  - Whether $u \to v$ is determined by the transition function of N.
- Hence reachability in G can be decided in $\mathrm{DSPACE}(s(n)^2)$.

We do not need to know $s(n)$ in advance, instead:

Let x be an input of length n.
For $s = \log n, \log n + 1, \ldots$

- If Path($\mathtt{start}$,$\mathtt{accept}$,s) where all intermediate configurations of N have size $\leq s$, then accept.
- If no configuration of size $s + 1$ is reachable from $\mathtt{start}$, then reject.

Since N runs in space $s(n)$, this loop will eventually halt. $\square$

The most prominent application of Savitch's Theorem is that deterministic and non-deterministic polynomial space coincide (same for exponential space).

Corollary

PSPACE=NPSPACE

An example in PSPACE

Membership for transformation semigroups

Input: transformations (functions) $a_1, \dots, a_k, b$ on $\{1, \dots, n\}$
Question: Is b generated by $a_1, \dots, a_k$ under composition, $b \in \langle a_1, \dots, a_k \rangle$?

Theorem (Kozen 1970s)

Membership for transformation semigroups is in PSPACE (actually PSPACE-complete).

Proof.

For $\ell \in \mathbb{N}$ let

$$A^\ell := \{a_{i_1} \dots a_{i_s} \ : \ s \leq \ell, i_1, \dots, i_s \in \{1, \dots, k\}\}.$$

$$A^0 \subseteq A^1 \subseteq A^2 \subseteq \dots$$

Since there are n^n functions on $\{1, \dots, n\}$, this chain stabilizes after $\leq n^n$ steps and $\langle a_1, \dots, a_k \rangle = A^{n^n}$.

Non-deterministic algorithm for membership:

1. Choose $c := a_{i_1}$ non-deterministically.
2. For $s = 2, \ldots, n^n$ do
3. Choose $i_s \in \{1, \ldots, k\}$ non-deterministically, let $c := ca_{i_s}$.
4. If $c = b$, then return `true`.
5. Return `false`.

Correctness: Every element in $\langle a_1, \ldots, a_k \rangle$ can be written as product of $\leq n^n$ generators.

Space complexity: $O(n)$ for storing, updating c.

Membership is in NPSPACE, hence in PSPACE by Savitch's Theorem. □