Incomparable degrees

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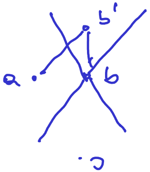
So far we constructed only degrees $\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \dots$

Question

Are Turing degrees linearly ordered?

Theorem (Avoiding cones)

For every degree $\mathbf{b} > \mathbf{0}$ there exists $\mathbf{0} < \mathbf{a} < \mathbf{b}'$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.



Proof.

B'-computable

Let B be non-computable. We want to construct A such that

- A is not computable, i.e.
  $R_{2e}: \ \chi_A \neq \varphi_e.$
- Whenever $C \leq_T A$ and $C \leq_T B$, then C is computable, i.e.
  $R_{2\sharp(e,f)+1}: \ \chi_C = \varphi_e^A = \varphi_f^B \ \Rightarrow C$ is computable.
  Here $\sharp$ denotes a computable bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

To satisfy these requirements for all $e, f \in \mathbb{N}$ we define initial segments σ_s of χ_A in stages.

Initialize $\sigma_0 := ()$.
Stage $s = 2e$**:** To satisfy R_{2e}, let $x := |\sigma_s|$. Put x into A iff $\varphi_e(x) = 0$ ($\emptyset'$-computable).

$$\sigma_{s+1} := \begin{cases} \sigma_s \circ (1) & \text{if } \varphi_e(x)\downarrow \text{ and } \varphi_e(x) = 0, \\ \sigma_s \circ (0) & \text{else.} \end{cases}$$

concatenation of tuples

Stage $s = 2\sharp(e, f) + 1$**:**

- Given σ_s test $\emptyset'$-computably whether

$$\exists \rho \; \exists \tau \; \exists x : \; \sigma_s \prec \rho, \tau, \; \varphi_e^{\rho}(x) \downarrow \neq \varphi_e^{\tau}(x) \downarrow \qquad (\dagger)$$

- **Case 1**, $(\dagger)$ holds (ρ, τ are e-**splitting** extensions of σ_s):
- Check B'-computably whether $\varphi_f^B(x) \neq \varphi_e^{\rho}(x)$.
  If yes, set $\sigma_{s+1} := \rho$; if no, $\sigma_{s+1} := \tau$.
- Then R_s holds since $\varphi_f^B \neq \varphi_e^A$.
- **Case 2**, $(\dagger)$ does not hold: Set $\sigma_{s+1} := \sigma_s$.
- **Claim:** If $\sigma_s \prec A$ and $\psi := \varphi_e^A$ is total, then ψ is computable.

  breadth first search
- To compute $\psi(x)$, dovetail $\varphi_e^{\tau}(x)$ for all τ extending σ_s.
- The first converging computation yields $\psi(x)$ (Since $(\dagger)$ does not hold, all converging computations yield the same result).
- Then R_s holds since $\chi_C = \varphi_e^A \Rightarrow C$ computable.

Finally

1. $\deg(A) > \mathbf{0}$ by requirements R_{2e},
2. $\deg(A) \wedge \deg(B) = \mathbf{0}$ by requirements $R_{2\sharp(e,f)+1}$,
3. $\deg(A) \leq \deg(B)'$ since A is computable in $\emptyset' \oplus B' \equiv_T B'$ by construction.

$\square$

The jump is onto degrees $\geq \mathbf{0'}$

- By the Jump Theorem 4, the jump is monotonous, i.e.

$$\mathbf{b} \leq \mathbf{a} \;\Rightarrow\; \mathbf{b}' \leq \mathbf{a}'.$$

- Hence $\mathbf{0}' \leq \mathbf{a}'$ for every degree $\mathbf{a}$.

Friedberg Completeness Criterion

For every degree $\mathbf{b} \geq \mathbf{0}'$, there is $\mathbf{a}$ such that $\mathbf{b} = \mathbf{a} \vee \mathbf{0}' = \mathbf{a}'$.

Proof

Let $B \subseteq \mathbb{N}$ with $\emptyset' \leq_T B$. Construct A such that

- $A' \leq_T B$,
- $B \leq_T A \oplus \emptyset'$

via finite initial segments σ_s $(s \in \mathbb{N})$ of A.

Initialize $\sigma_0 := ()$.

Stage $s = 2e$ (Decide whether $e \in A'$):
Given σ_s test $\emptyset'$-computably whether

$$\exists \tau : \ \sigma_s \prec \tau, \ \varphi_e^{\tau}(e) \downarrow$$

Let σ_{s+1} be the smallest such τ if it exists; else $\sigma_{s+1} := \sigma_s$

Stage $s = 2e + 1$ (Code $\chi_B(e)$ into A): $\sigma_{s+1} := \sigma_s \circ (\chi_B(e))$

Let $A \subseteq \mathbb{N}$ with characteristic function $\bigcup_{s \in \mathbb{N}} \sigma_s$ (limit of the σ_s).

1. σ_s and hence A is computable in $B \equiv_T \emptyset' \oplus B$ by construction.
2. $A' \leq_T B$ since

   $$e \in A' \text{ iff } \varphi_e^{\sigma_{2e+1}}(e) \downarrow$$

   where σ_{2e+1} is B-computable by 1.
3. σ_s is computable in $A \oplus \emptyset'$ by induction on s:
   - Given σ_{2e}, compute σ_{2e+1} using $\emptyset'$-oracle.
   - Given σ_{2e+1}, compute $\sigma_{2e+2} = \sigma_{2e+1} \circ (\chi_A(|\sigma_{2e+1}|))$ with A-oracle.
4. $B \leq_T A \oplus \emptyset'$ by 3. since $\chi_B(e)$ is the last entry in σ_{2e+2}. $\square$

The jump is not injective

Theorem (Spector (1956))
There exists a degree $\mathbf{a} > \mathbf{0}$ such that $\mathbf{a}' = \mathbf{0}'$.

Proof.
We want to construct $A \subseteq \mathbb{N}$ such that

- A is not computable, $R_{2e} : \ \chi_A \neq \varphi_e.$
- $A' \leq \emptyset'$ $R_{2e+1} : \ \varphi_e^A(e) \downarrow$ is $\emptyset'$-computable.

To satisfy these requirements for all $e \in \mathbb{N}$ define finite initial segments σ_s of A in stages.

Initialize $\sigma_0 := ()$.
Stage $s = 2e$ (Put x into A iff $\varphi_e(x) = 0$):
Let $x := |\sigma_s|$.

$$\sigma_{s+1} := \begin{cases} \sigma_s \circ (1) & \text{if } \varphi_e(x)\downarrow \text{ and } \varphi_e(x) = 0, \\ \sigma_s \circ (0) & \text{else.} \end{cases}$$

Stage $s = 2e + 1$ (Decide whether $e \in A'$):
Given σ_s test $\emptyset'$-computably whether

$$\exists \tau : \ \sigma_s \prec \tau, \ \varphi_e^\tau(e)\downarrow$$

Let σ_{s+1} be the smallest such τ if it exists; else $\sigma_{s+1} := \sigma_s$.

Let $A \subseteq \mathbb{N}$ with characteristic function $\bigcup_{s \in \mathbb{N}} \sigma_s$. Then

1. σ_s and hence A is computable in $\emptyset'$ by construction.
2. $A' \leq_T \emptyset'$ since
$$e \in A' \text{ iff } \varphi_e^{\sigma_{2e+2}}(e)\downarrow$$
where σ_{2e+2} is $\emptyset'$-computable by 1. $\square$