Arithmetical hierarchy and Turing jumps

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Question

What is the connection between

- the arithmetical hierarchy (classification of sets by definability)
- and Turing degrees (classification by computability)?

Finite approximations

Analyzing oracle machines requires computable approximations:
If $\varphi_e^A(x)\downarrow$, then only a finite part of A is used in this computation.

- For $A \subseteq \mathbb{N}$ and $s \in \mathbb{N}$, the s-tuple

$$\sigma := (\chi_A(0), \chi_A(1), \ldots, \chi_A(s-1)) \in \{0,1\}^s$$

  is an **initial segment** (finite approximation) of χ_A, denoted $\sigma \prec A$.
- For $\sigma \in \{0,1\}^s$ write
  $|\sigma| = s$,
  $\sigma = (\sigma(0), \ldots, \sigma(s-1))$ and
  $\sharp(\sigma) := \prod_{i<|s|} p_i^{\sigma(i)+1}$ (prime power encoding).
  Note that $|\sigma|, \sigma(i)$ are computable from $\sharp(\sigma), i$.
- For $\sigma, \tau \in \{0,1\}^*$ write $\sigma \prec \tau$ and call σ an initial segment of τ if $|\sigma| \leq |\tau|$ and $\sigma(i) = \tau(i)$ for all $i \leq |\sigma|$.

Definition

- Let $A \subseteq \mathbb{N}$. If M_e^A halts on input x with output y and if u is the maximum element for which the oracle is used (queried for $u \in A$) during the computation, write

$$\varphi_e^A(x) := y \qquad \mathrm{use}_e^A(x) := u.$$

  use_e^A is called the **use function** corresponding to $\varphi_e^A(x)$.

- $\varphi_e^\sigma(x) := y$ if $\varphi_e^A(x) = y$, $\sigma \in \{0,1\}^*$ with $\sigma \prec A$ and $\mathrm{use}_e^A(x) < |\sigma|$ (i.e., only σ is queried).
- $\varphi_{e,s}^A(x) := y$ if $\varphi_e^A(x) = y$ is computed by M_e^A in $< s$ steps and $e, x, y, \mathrm{use}_e^A(x) < s$.
- $\varphi_{e,s}^\sigma(x) := y$ if $\varphi_{e,s}^A(x) = y$, $\sigma \in \{0,1\}^*$ with $\sigma \prec A$ and $\mathrm{use}_e^A(x) < |\sigma|$.
- $W_{e,s}^\sigma := \mathrm{domain}\, \varphi_{e,s}^\sigma$, etc.

Computable approximations

Lemma

1. $\varphi_e^A(x) = y$ iff $\exists s\ \exists \sigma \prec A:\ \varphi_{e,s}^\sigma(x) = y$
2. If $\varphi_{e,s}^\sigma(x) = y$, then $\forall t \geq s\ \forall \tau \mathrm{succ} \sigma:\ \varphi_{e,t}^\tau(x) = y$.
3. $W_{e,s}^\sigma$ (i.e. $\{(e, \sharp(\sigma), x, s)\ :\ \varphi_{e,s}^\sigma(x)\downarrow\}$) is computable.

Proof.

1. Any computation that halts, does so after finitely many steps, using a finite part of the oracle.
2. If a computation halts after s steps with access to σ, its output will not change when given more time and a larger part of the oracle.
3. Run M_e on x with queries to σ until it halts or s steps are completed.

$\square$

Post's Theorem relating Σ_n and $\emptyset^{(n)}$

Recall

- $B \subseteq \mathbb{N}$ is Σ_n if there is some computable $R \subseteq \mathbb{N}^{n+1}$ such that $B = \{x \ : \ \exists y_1 \ \forall y_2 \ldots \exists/\forall y_n \ (x, y_1, \ldots, y_n) \in R\}$.
- $A' := \{x \ : \ \varphi_x^A(x) \downarrow\}$.

Post's Theorem

Let $n \in \mathbb{N}, B \subseteq \mathbb{N}$.

1. B is Σ_{n+1} iff B is c.e. in some Π_n-set iff B is c.e. in some Σ_n-set.
2. $\emptyset^{(n)}$ is Σ_n-complete for $n > 0$.
3. B is Σ_{n+1} iff B is $\emptyset^{(n)}$-c.e.
4. B is Δ_{n+1} iff $B \leq_T \emptyset^{(n)}$.

Σ_{n+1} = c.e in $\phi^{(n)}$

Δ_{n+1} = computable in $\phi^{(n)}$

Σ_n = many-one reducible to $\phi^{(n)}$

Note: Properties of Σ_1 relativize to $\Sigma_{n+1} = \Sigma_1^{\emptyset^{(n)}}$ by 3.

Proof 1.

$\Rightarrow$: Let $B \in \Sigma_{n+1}$. Then we have $R \in \Pi_n$ such that

$$x \in B \text{ iff } \exists y\ R(x,y).$$

Then $B \in \Sigma_1^R$, hence R-c.e.

$\Leftarrow$: Assume B is A-c.e. for some $A \in \Pi_n$. Then for some e

$$\begin{aligned} x \in B &\text{ iff } x \in W_e^A \\ &\text{ iff } \exists s\, \exists \sigma \in \{0,1\}^* :\ \sigma \prec A \wedge \underbrace{x \in W_{e,s}^{\sigma}}_{\text{computable}} \end{aligned}$$

Claim: $\sigma \prec A$ is Σ_{n+1}

$$\begin{aligned} \sigma \prec A &\text{ iff } \forall y \leq |\sigma| :\ \sigma(y) = \chi_A(y) \\ &\text{ iff } \forall y \leq |\sigma| : \underbrace{\underbrace{(\sigma(y) = 1, y \in A)}_{\Pi_n} \vee \underbrace{(\sigma(y) = 0, y \notin A)}_{\Sigma_n}}_{\Sigma_{n+1}} \end{aligned}$$

Since Σ_{n+1} is closed under bounded $\forall$, the claim follows.
Then $B \in \Sigma_{n+1}$.

Note: A-c.e $= \bar{A}$-c.e. yields the second equivalence in 1.

Proof

2. Show $\emptyset^{(n)}$ is Σ_n-complete for $n > 0$ by induction.

Base case: For $n = 1$, $\emptyset' = K$ is Σ_1-complete.

Induction step: Let $B \subseteq \mathbb{N}$. Then

$$\begin{aligned} B \in \Sigma_{n+1} & \text{ iff } B \text{ is c.e. in some } \Sigma_n \text{ set by 1.} \\ & \text{ iff } B \text{ is c.e. in } \emptyset^{(n)} \text{ by induction assumption} \\ & \text{ iff } B \leq_m \emptyset^{(n+1)} \text{ by the Jump Theorem 2.} \end{aligned}$$

Hence $\emptyset^{(n+1)}$ is Σ_{n+1}-complete.

3. follows from 1. and 2.

4. $$\begin{aligned} B \in \Delta_{n+1} & \text{ iff } B, \bar{B} \in \Sigma_{n+1} \\ & \text{ iff } B, \bar{B} \text{ are } \emptyset^{(n)}\text{-c.e. by 3.} \\ & \text{ iff } B \text{ is } \emptyset^{(n)}\text{-computable by Complementation Thm.} \end{aligned}$$

□