

Turing degrees

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Turing reductions

A, B denote subsets of \mathbb{N} throughout.

Definition

B is **Turing reducible** to A (denoted $B \leq_T A$) if B is computable in A .

Many-one vs Turing reduction

- ▶ $B \leq_m A$: \exists computable f such that $\chi_B = \chi_A \circ f$
Oracle A is queried only once after encoding x as $f(x)$.
- ▶ $B \leq_T A$: χ_B is recursive in χ_A
Oracle χ_A can be used several times in defining χ_B .

Turing reductions are the most general computable reductions from B to A .

Definition

B is **computably enumerable in A** if $B = W_e^A$ for some e where $W_e^A = \text{domain } \varphi_e^A$.

Our current theory for c.e. sets relativizes to A -c.e. sets.

Relativized Complementation Theorem

$B \leq_T A$ iff B and \bar{B} are c.e. in A .

Proof: HW

Theorem

TFAE:

1. B is c.e. in A ;
2. $B = \emptyset$ or B is the range of some A -computable total function;
3. B is Σ_1^A .

Turing degrees

Definition

- ▶ A is **Turing equivalent** to B (denoted $A \equiv_T B$) if $A \leq_T B$ and $B \leq_T A$.
- ▶ \equiv_T is an equivalence relation on the subsets of \mathbb{N} .
- ▶ The **Turing degree** of A is the equivalence class

$$\deg(A) := \{B : B \equiv_T A\}$$

- ▶ A degree is c.e. if it contains a c.e. set.
- ▶ The set of all degrees \mathbf{D} is partially ordered by

$$\deg(A) \leq \deg(B) \text{ iff } A \leq_T B.$$

Basic order properties

1. The smallest degree in \mathbf{D} is

$$\mathbf{0} := \deg(\emptyset) = \{A : A \text{ is computable}\}$$

2. Any two degrees have a **supremum** (least upper bound)

$$\deg(A) \vee \deg(B) = \deg(A \oplus B)$$

where

$$A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$$

encodes the disjoint union of A and B .

(HW)

3. The supremum of infinitely many degrees and the **infimum** (greatest lower bound) of two degrees need not always exist. (postponed)
4. \mathbf{D} is a \vee -semilattice, not a lattice.

Turing jump

Problem

Given a degree $\mathbf{a} = \deg(A)$ find a degree $\mathbf{b} = \deg(B)$ that is strictly greater than \mathbf{a} (denoted $\mathbf{a} < \mathbf{b}$).

In other words, find B that is not A -computable.

Definition

For $A \subseteq \mathbb{N}$, the relativized diagonal Halting Problem

$$A' := K^A = \{x : \varphi_x^A(x) \downarrow\} = \{x : x \in W_x^A\}$$

is the **jump** of A .

Basic properties of the jump

Jump Theorem

For $A, B \subseteq \mathbb{N}$

1. A' is A -c.e.;
2. B is A -c.e. iff $B \leq_m A'$;
3. $A <_T A'$;
4. $B \leq_T A$ iff $B' \leq_m A'$;
5. ~~If $B \equiv_T A$, then $B' \equiv_m A'$.~~

iff

Simplified for $A = \emptyset$

K is c.e.

B is c.e. iff $B \leq_m K$

Proof 1.

By the Relativized Enumeration Theorem we have z such that

$$\forall e, x : \varphi_e^A(x) = \varphi_z^A(e, x).$$

Then $\psi(x) := \varphi_z^A(x, x)$ is A -computable and $A' = \text{domain } \psi$ is A -c.e. □

Proof 2.

\Rightarrow : Let $B = \text{domain } \varphi_e^A$ for some e . By the Relativized S_m^n -Theorem we have a computable s such that

$$\forall x, y : \varphi_e^A(x) = \varphi_{s(x)}^A(y).$$

Then $x \in B$ iff $\varphi_e^A(x) \downarrow$
iff $\varphi_{s(x)}^A(s(x)) \downarrow$
iff $s(x) \in A'$.

Hence s is a many-one reduction from B to A' .

\Leftarrow : Let s be a many-one reduction from B to A' .

Then $x \in B$ iff $\varphi_{s(x)}^A(s(x)) \downarrow$
iff $\varphi_e^A(x) \downarrow$ (where $e \in \mathbb{N}$ is determined by s)

Hence $B = \text{domain } \varphi_e^A$ is A -c.e. □

Proof 3, 4.

(HW)

Jump of degrees

Definition

For the degree \mathbf{a} the jump

$$\mathbf{a}' := \deg(A') \text{ for some } A \in \mathbf{a}$$

is welldefined by the Jump Theorem 5.

Hence there is an infinitely ascending chain of degrees:

$$\mathbf{0} = \deg(\emptyset) = \{A : A \text{ is computable} \}$$

$$\mathbf{0}' = \deg(\emptyset') = \deg(K) \quad \text{by definition}$$

$$\mathbf{0}'' = \deg(\emptyset'') = \deg(F) = \deg(T) \quad \text{later}$$



Outlook

Question

- ▶ **Post's Problem:** Is there a c.e. degree other than $\mathbf{0}$ and $\mathbf{0}'$?
- ▶ Are there incomparable degrees?
- ▶ Does the infimum exists for any two degrees?