Approximations and the Friedberg Splitting Theorem

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Computability Theory, October 20, 2023

Computable approximation of sets

Definition

- $\varphi_{e,s}(x) := y$ if $e, x, y < s$ and the DTM M_e computes $\varphi_e(x) = y$ in $< s$ steps.
- If such y exists, say $\varphi_{e,s}(x)$ **converges** and write $\varphi_{e,s}(x)\downarrow$; else $\varphi_{e,s}(x)$ **diverges** and $\varphi_{e,s}(x)\uparrow$.
- $W_{e,s} := \mathrm{domain}\,\varphi_{e,s}$

Note

- $\varphi_e(x) = y$ iff $\exists s\ \varphi_{e,s}(x) = y$.
- If $x \in W_{e,s}$, then $x, e < s$.
- If $s < t$, then $W_{e,s} \subseteq W_{e,t}$.
- $W_e = \bigcup_{s\in\mathbb{N}} W_{e,s}$

Lemma

The following predicates are computable:

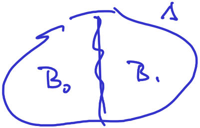
1. $\{(e, x, y, s) \ : \ \varphi_{e,s}(x) = y\}$
2. $\{(e, x, s) \ : \ \varphi_{e,s}(x) \downarrow\}$
3. $W_{e,s}$ (finite)

Proof.

Compute $\varphi_e(x)$ until some output is found or s steps are completed. □

A property of c.e. sets W_e is **dynamic** if it is described in terms of $W_{e,s}$ (time dependent).
So far most properties were **static** (e.g. lattice theoretic).

A static result with dynamic proof



Friedberg Splitting Theorem

Let $A \subseteq \mathbb{N}$ be c.e., noncomputable. Then there exist c.e. B_0, B_1 such that

$$A = B_0 \cup B_1, \; B_0 \cap B_1 = \emptyset, \text{ and } B_0, B_1 \text{ are computably inseparable.}$$

In particular B_0, B_1 are noncomputable.

Proof.

Enumerate A and put elements into B_0, B_1 to meet **requirements**

$$R_{e,i} : \qquad W_e \cap B_i \neq \emptyset$$

for $e \in \mathbb{N}, i \in \{0, 1\}$ if possible (Then B_i cannot be computable). At each stage try to satisfy $R_{e,i}$ of **highest priority** (smallest e) that does not hold yet.

Let $f: \mathbb{N} \to \mathbb{N}$ be injective, computable with $f(\mathbb{N}) = A$.

Stage s=0: $B_{0,0} := B_{1,0} := \emptyset$

Stage s+1: Let $e < s$ and $i \in \{0, 1\}$ be minimal such that

$$f(s) \in W_{e,s} \text{ and } W_{e,s} \cap B_{i,s} = \emptyset.$$

Set

$$B_{i,s+1} := B_{i,s} \cup \{f(s)\} \text{ and } B_{1-i,s+1} := B_{1-i,s}$$

Then $R_{e,i}$ received attention and remains satisfied forever.
If no such e, i exist, put $f(s)$ into $B_{0,s+1}$.

By construction

$$B_i := \bigcup_{s \in \mathbb{N}} B_{i,s}, \quad i \in \{0, 1\}$$

is c.e., B_0, B_1 are disjoint and $B_0 \cup B_1 = A$.

since f is injective

It remains to show: B_0, B_1 are computably inseparable.

Seeking a contradiction, suppose there is a computable C with

$$B_0 \subseteq C, \ B_1 \cap C = \emptyset.$$

For $C = W_e, \bar{C} = W_d$,

$$W_{d,s} \cap B_{0,s} = \emptyset \text{ and } W_{e,s} \cap B_{1,s} = \emptyset \ \ \forall s \in \mathbb{N}.$$

Still $R_{d,0}$ and $R_{e,1}$ never received attention. Why not?

- e_s in the construction above takes no value more than twice. Hence $\exists N \, \forall s > N : \ e_s > e, d$.
- $f(s) \notin W_{d,s}$ for $s > N$ because else we'd put $f(s) \in B_{0,s+1}$ and $R_{d,0}$ received attention instead of $R_{e_s,i}$ at stage $s+1$.
- Similar $f(s) \notin W_{e,s}$ for any $s > N$.

Hence

$$f(s) \notin W_{e,s} \cup W_{d,s} \quad \forall s > N \qquad (\dagger)$$

Claim: $\bar{A} = \bigcup_{s>N}(W_{e,s} \cup W_{d,s}) \setminus \{f(0), \ldots, f(s-1)\}$

- $\supseteq$: Clearly $f(0), \ldots, f(N)$ is not in the set on the right. Suppose $f(t) \in W_{e,s} \cup W_{d,s}$ for $t \geq s > N$. Then $f(t) \in W_{e,t} \cup W_{d,t}$ contradicts (†).
- $\subseteq$: Since $W_e \cup W_d = \mathbb{N}$, every $x \in \bar{A}$ occurs in some $(W_{e,s} \cup W_{d,s}) \setminus \{f(0), \ldots, f(s-1)\}$ for $s > N$.

By this claim $\bar{A}$ is c.e. contradicting the assumption that A is not computable.
Thus there are no e, d as above and B_0, B_1 are computably inseparable. □

Note

The proof is based on a simultanous enumeration of all c.e. sets to construct $B_{i,s}$.
By (†) $f(s)$ appears in A "earlier" than in W_e or W_d.