Productive and creative sets

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Computability Theory, October 13, 2023

So far all naturally occurring c.e. sets that are not computable (halting problem, acceptance problem) are Σ^0_1-complete with respect to many-one reductions.

Question

Is every non-computable c.e. set many-one complete for Σ^0_1?

The answer requires quite a bit of preparation.

Recursion Theorem

Theorem (Kleene)

For every computable function f there exists $n \in \mathbb{N}$ (a "fixed point" of f) such that

$$\varphi_n = \varphi_{f(n)}.$$

We prove a stronger version on the next page.

Corollary

domain φ_n

There exists $n \in \mathbb{N}$ such that $W_n = \{n\}$.

Proof.

- The partial function $\psi(x, y) := x$ if $x = y$ is computable.
- By the s-m-n Theorem there exists a computable f such that $\psi(x, y) = \varphi_{f(x)}(y)$. So $W_{f(x)} = \{x\}$.
- By the Recursion Theorem there exists n such that $W_n = W_{f(n)} = \{n\}$.

Uniform Recursion Theorem (Kleene)

For every computable $f(x, y)$ there exists a computable $h(y)$ such that

$$\forall y, z: \ \varphi_{h(y)}(z) = \varphi_{f(h(y),y)}(z)$$

Proof.

- Define a computable d by

.
$$\varphi_{d(x,y)}(z) := \begin{cases} \varphi_{\varphi_x(x,y)}(z) & \text{if } \varphi_x(x,y)\downarrow, \\ \text{undefined} & \text{else.} \end{cases}$$

- Then $f(d(x,y), y) \overset{(*)}{=} \varphi_v(x, y)$ for some v.
- $h(y) := d(v, y)$ is the required fixed point since

$$\varphi_{\underline{d(v,y)}} \overset{\text{Def of } d}{=} \varphi_{\varphi_v(v,y)} \overset{(*)}{=} \varphi_{f(\underline{d(v,y)}),y)}.$$

$\square$

Creative sets

Question

How to show a c.e. set $A \subseteq \mathbb{N}$ is not computable?

Recall: Every c.e. A is of the form $W_x := \mathrm{domain}\, \varphi_x^{(1)}$.

A is **not** computable iff $\bar{A}$ is not c.e.
iff $\forall x\ \bar{A} \neq W_x$
iff $\forall x\ [W_x \subseteq \bar{A} \Rightarrow \exists y\ y \in \bar{A} \setminus W_x]$

Definition

$B \subseteq \mathbb{N}$ is **productive** if there exists a total computable function p:

$$\forall x\ W_x \subseteq B \Rightarrow p(x) \in B \setminus W_x$$

$A \subseteq \mathbb{N}$ is **creative** if A is c.e. and $\bar{A}$ is productive.

Note

productive = p produces witnesses for not being c.e.

creative = "effectively" non-computable

c.e &

Example

The complement of $K = \{x \ : \ \varphi_x(x)\downarrow\}$ is productive. with $p(x) = x$.
Assume $W_x \subseteq \bar{K}$.

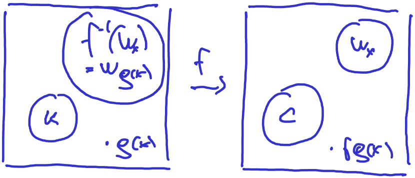
- Then $x \notin W_x$ since otherwise $\varphi_x(x)\downarrow$ yields $x \in K$.
- Further $x \notin K$. Hence $x \in \bar{K} \setminus W_x$.

Thus K is creative.

Goal

Show Σ^0_1-complete $=$ creative.

Σ^0_1-complete $\Rightarrow$ creative



Theorem

Every Σ^0_1-complete set C is creative.

Proof.

Assume $K \leq_m C$ via a total computable f, i.e.,

$$x \in K \text{ iff } f(x) \in C$$

It remains to show that $\bar{C}$ is productive.

- Assume $W_x \subseteq \bar{C}$ for some x. Then $f^{-1}(W_x) \subseteq \bar{K}$.
- Note there exists a total computable g such that

$$\varphi_x(f(y)) = \varphi_{g(x)}(y) \quad \forall y.$$

- Then $W_{g(x)} = f^{-1}(W_x)$.
- $W_{g(x)} \subseteq \bar{K}$ implies $g(x) \in \bar{K} \setminus W_{g(x)}$ as in the previous ex.
- Thus $fg(x) \in f(\bar{K}) \subseteq \bar{C}$ and $fg(x) \notin W_x$.

So $\bar{C}$ is productive via fg.

Creative $\Rightarrow \Sigma_1^0$-complete

Theorem (Myhill, 1955)

A set is creative iff it is Σ_1^0-complete.

Proof

$\Leftarrow$: already done

$\Rightarrow$: Let C be creative and p total computable such that

$$\forall x \; W_x \subseteq \bar{C} \Rightarrow p(x) \in \bar{C} \setminus W_x$$

Let A be Σ_1^0. We construct a many-one reduction ph from A to C: Consider

$$\varphi_{f(x,y)}(z) := \begin{cases} 1 & \text{if } z = p(x), y \in A, \\ \text{undefined} & \text{else.} \end{cases}$$

By the Uniform Recursion Theorem, we have computable h such that

$$\varphi_{h(y)}(z) = \varphi_{f(h(y),y)}(z) \quad \forall y, \cancel{x}\, z.$$

Then

$$W_{h(y)} = \begin{cases} \{ph(y)\} & \text{if } y \in A \\ \emptyset & \text{else.} \end{cases}$$

Claim: ph is a many-one reduction from A to C.

1. Assume $y \notin A$: Then $W_{h(y)} = \emptyset \subseteq \bar{C}$ yields $ph(y) \in \bar{C} \setminus \emptyset$.
2. Assume $y \in A$: Then $W_{h(y)} = \{ph(y)\} \not\subseteq \bar{C}$ because otherwise $ph(y) \in \bar{C} \setminus \{ph(y)\}$. Hence $ph(y) \in C$.

$\square$

All Σ^0_1-complete sets are isomorphic via computable functions

Theorem (Myhill)

For every creative set C there exists a computable bijection $f : \mathbb{N} \to \mathbb{N}$ such that

$$C = f(K)$$

Proof.

Omitted. $\square$