Many-one completeness for arithmetical hierarchy

Peter Mayr

Computability Theory, October 9, 2023

What are the hardest Σ^0_n-problems?

To simplify notation we only consider subsets of $\mathbb{N}$.

Recall

- For $A, B \subseteq \mathbb{N}$, A is **many-one reducible** to B (short $A \leq_m B$) if there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$:

$$\forall x \in \mathbb{N} : x \in A \text{ iff } f(x) \in B.$$

- A is c.e. iff $A \leq_m AP$ (HW). Hence the acceptance problem is "hardest" among Σ^0_1-sets.

Question

Can this be generalized to higher levels of the arithmetical hierarchy?

Closure under many-one reductions

Lemma

If $A \leq_m B$ and B is Σ^0_n, then A is Σ^0_n (dually for Π^0_n).

Proof.

Assume $f : A \to B$ is a many-one reduction and $B(z)$ is Σ^0_n. Then

$$A(x) \equiv B(f(x))$$

is Σ^0_n since Σ^0_n is closed under substitution by total computable functions. □

Σ^0_n-complete sets

Definition

$C \subseteq \mathbb{N}$ is Σ^0_n-**complete** if

1. C is Σ^0_n and
2. for every Σ^0_n-set A: $A \leq_m C$.

Theorem

For each $n \geq 1$

1. Σ^0_n-complete sets exist;
2. no Σ^0_n-complete set is Π^0_n.

Universal $\Rightarrow$ complete

Proof.

1. A universal Σ^0_n-predicate $U_n(e, x)$ is Σ^0_n-complete since for each A in Σ^0_n, we have $e \in \mathbb{N}$:

$$A(x) \text{ iff } U_n(e, x).$$

f(x) total computable $\mathbb{N} \to \mathbb{N}^2$

2. Recall: $K_n(x) = U_n(x, x)$ is Σ^0_n, not Π^0_n.
Let C be Σ^0_n-complete.
Then $K_n \leq_m C$ and C cannot be Π^0_n either. □

Further complete examples 1

$T = \{e : \varphi_e \text{ is total}\}$ is Π^0_2-complete.

Proof.

Recall T is Π^0_2. Let R be computable and

$e \in T$ iff $\forall x \exists n\ \varphi_{e,n}(x)\downarrow$

$$A(x) \equiv \forall y \exists z\ R(x,y,z) \qquad (\Pi^0_2)$$

- Define $\psi(x,y) := \mu z\ R(x,y,z)$.
- By the S^m_n-Theorem for $m = n = 1$, we have a computable $h := S^1_1$ such that

$$\psi(x,y) = \varphi_{h(x)}(y) \text{ for all } x, y.$$

- Then $x \in A$ iff $\forall y\ \varphi_{h(x)}(y)\downarrow$
  iff $\varphi_{h(x)}$ is total
  iff $h(x) \in T$.
- Hence the S^m_n-Theorem yields a many-one reduction h from A to T.

Further complete examples 2, 3

The **diagonal halting problem** $K = \{x \ : \ \varphi_x(x)\downarrow\}$ is Σ^0_1-complete.

Proof

Let R be computable and

$$A(x) \equiv \exists y \ R(x, y) \qquad\qquad (\Sigma^0_1)$$

- Define $\psi(x, z) := \mu y \ R(x, y)$ (independent of z!).
- By the S^m_n-Theorem, we have a computable h such that

$$\psi(x, z) = \varphi_{h(x)}(z) \text{ for all } x, z.$$

- Then $x \in A$ iff $\psi(x, z)\downarrow$
  iff $\varphi_{h(x)}(h(x))\downarrow$
  iff $h(x) \in K$. $\square$

$K_n := \{x \ : \ U_n(x, x)\}$ is Σ^0_n-complete for any $n \geq 1$.