Arithmetical Hierarchy

Peter Mayr

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DTM vs functions on $\mathbb{N}$

For a partial function f write

- $f(x)\downarrow$ if x is in the domain of f;
- $f(x)\uparrow$ if x is not in the domain of f.

$\varphi_e(x)\colon \mathbb{N} \to_p \mathbb{N}$ is computed by the DTM with Goedel number e

Facts

- $A \subseteq \mathbb{N}^k$ is computably enumerable iff A is the domain of some partial recursive function.
- $A \subseteq \mathbb{N}^k$ is computable iff the characteristic function of A is recursive.
- The **Diagonal Halting Problem**

$$K := \{x \in \mathbb{N} \ : \ \varphi_x(x)\downarrow\}$$

is c.e. but not computable.

Properties of recursive functions are not computable

Rice's Theorem

Let C be a class of k-ary recursive functions. Then $\{e \in \mathbb{N} \ : \ \varphi_e \in C\}$ is computable iff $C = \emptyset$ or C is the class of all k-ary recursive functions.

Example

None of the following are computable:

- $K := \{x \in \mathbb{N} \ : \ \varphi_x(x) \downarrow\}$
- $F := \{x \in \mathbb{N} \ : \ \varphi_x \text{ has finite domain}\}$
- $T := \{x \in \mathbb{N} \ : \ \varphi_x \text{ is total }\}$

The arithmetical hierarchy of subsets of $\mathbb{N}$

Idea: Classify problems that are not computable by the complexity of formulas that describe them.

Example (Diagonal halting problem K)

$$x \in K \text{ iff } \varphi_x(x)\downarrow \qquad \Sigma^0_1$$

$$\text{iff } \exists y \; \underbrace{(\mathrm{config}(x,x,y))_0 = t}_{\substack{\text{computable predicate} \\ \text{stating computation of } \varphi_x \text{ on } x \\ \text{halts after y steps}}}$$

Definition

Let $P(\bar{x})$ be a k-ary predicate on $\mathbb{N}$, $n \in \mathbb{N}$:

- P is Σ^0_n if there is a computable predicate R:

$$P(\bar{x}) \equiv \underbrace{\exists y_1 \forall y_2 \exists y_3 \ldots \exists/\forall y_n}_{n \text{ alternating quantifiers starting with } \exists} : R(\bar{x}, \bar{y})$$

$= (y_1, \ldots, y_n)$

- P is Π^0_n if there is a computable predicate R:

$$P(\bar{x}) \equiv \underbrace{\forall y_1 \exists y_2 \forall y_3 \ldots \exists/\forall y_n}_{n \text{ alternating quantifiers starting with } \forall} : R(\bar{x}, \bar{y})$$

- $\Sigma^0_0 = \Pi^0_0 =$ computable predicates
- $\Delta^0_n := \Sigma^0_n \cap \Pi^0_n$

Note

The superscript 0 denotes quantification over type-0-objects (elements in $\mathbb{N}$).

Example

1. K is Σ^0_1
2. $T = \{e \in \mathbb{N} \ : \ \varphi_e \text{ is total}\}$

   $e \in T$ iff $\forall x \ \varphi_e(x) \downarrow$

   iff $\forall x \, \exists y \ (\mathrm{config}(e, x, y))_0 = t$

   Hence T is Π^0_2.
3. $F = \{e \in \mathbb{N} \ : \ \varphi_e \text{ has finite domain}\}$

   $e \in F$ iff $\exists z \forall y \, \forall x^{*} \ (\mathrm{config}(e, x, y))_0 = t \Rightarrow x \leq z$

   $(u)_0 \ (u)_1$

   Hence F is Σ^0_2.

   $* \ \forall u$

Arithmetical hierarchy

$\bigcup_{n\in\mathbb{N}} \Sigma^0_n = \bigcup_{n\in\mathbb{N}} \Pi^0_n$ (sets defined in first order arithmetic, hence called **arithmetical**)

$$\begin{tikzcd}[column sep=small]
& \vdots & \\
\Sigma^0_2 \arrow[dr, dash] \arrow[ur, dash] & & \Pi^0_2 \arrow[dl, dash] \arrow[ul, dash] \\
& \Delta^0_2 & \\
\Sigma^0_1 \arrow[ur, dash] \arrow[dr, dash] & & \Pi^0_1 \arrow[ul, dash] \arrow[dl, dash] \\
& \Delta^0_1 &
\end{tikzcd}$$

$\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2 \supseteq \Sigma^0_1 \cup \Pi^0_1$

c.e. Σ^0_1 $\quad$ Π^0_1 co c.e

computable $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1 = \Sigma^0_0 \cup \Pi^0_0$

Question

Are all subsets of $\mathbb{N}$ arithmetical?

Closure properties

For proving the previous picture we need some preparation.

Lemma

Let $n \geq 1$.

1. Σ_n^0 is closed under existential quantification, Π_n^0 is closed under universal quantification.
2. Σ_n^0, Π_n^0 are both closed under $\wedge, \vee$, bounded quantifiers $\forall x < y, \exists x < y$, and substitution of total computable functions.
3. $\neg\Sigma_n^0 = \Pi_n^0, \neg\Pi_n^0 = \Sigma_n^0$

Proof sketch.

Let R be computable, $P(x,z) = \exists y_1 \forall y_2 \ldots \; R(x,z,y_1,\ldots)$ be Σ_n^0.

1. **Claim:** $Q(x) := \exists z \; P(x,z)$ is Σ_n^0

   $$\begin{aligned} Q(x) &\equiv \exists z \exists y_1 \forall y_2 \ldots \; R(x,z,y_1,y_2,\ldots,y_n) \\ &\equiv \quad \exists u \quad \forall y_2 \ldots \; R(x,(u)_0,(u)_1,y_2\ldots,y_n) \end{aligned}$$

   Dual argument for $\forall$ and Π_n^0.

2. **Substitution:** Let $f(x)$ total, computable.

   Claim: $Q(x) := P(x, f(x))$ is Σ_n^0

   $$Q(x) \equiv \exists y_1 \forall y_2 \ldots \; \underbrace{R(x,f(x),y_1,y_2,\ldots)}_{\text{computable since } R \text{ is}}$$

   Conjunction: Induct on n (HW).

3. **Negation:** immediate. de Morgan

$\square$

Σ^0_1 is computably enumerable

Normal Form Theorem for c.e. sets
P is c.e. iff P is Σ^0_1.

Proof

$\Rightarrow$: Let $P \subseteq \mathbb{N}^k$ be c.e.

- Then $P = \mathrm{domain}\, \varphi_e^{(k)}$ for some e (HW).
- $x \in P$ iff $\varphi_e(x)\downarrow$

  iff $\exists n \; \underbrace{(\mathrm{config}(e, x, n))_0 = t}_{=:\varphi_{e,n}(x)\downarrow}$ universal Σ^0_1

  computes in n steps
- P is Σ^0_1 because the predicate $\varphi_{e,n}(x)\downarrow$ ("M_e computes $\varphi_e(x)$ in n steps") is computable.

$\Leftarrow$: Let $P(x) \equiv \exists y \; R(x, y)$ for R computable.

- Then $\psi(x) := \mu y\, R(x, y)$ is recursive.
- $P = \mathrm{domain}\, \psi$ is c.e. $\square$

Dually Π^0_1 is co-c.e.

Universal Σ^0_n predicates

Idea: Enumerate k-ary Σ^0_n predicates by a single $k+1$-ary Σ^0_n predicate.

Definition

A $k+1$-ary predicate $U(e, \bar{x})$ is **universal Σ^0_n** for k-ary predicates if

1. $U(e, \bar{x})$ is Σ^0_n and
2. for every k-ary Σ^0_n predicate $P(\bar{x})$ there is some e such that

$$P(\bar{x}) \equiv U(e, \bar{x}).$$

Universal Π^0_n-predicates are defined correspondingly.

Example

From the last proof $U(e, x) := \exists n \; \varphi_{e,n}(x) \downarrow$ is universal Σ^0_1.

Enumeration Theorem

For all $k, n \geq 1$, universal Σ^0_n- and Π^0_n-predicates exist.

Proof by induction on n ~~and k:~~

Base case: $U(e,x) := \exists m\ \varphi_{e,m}(x)\downarrow$ is universal Σ^0_1 by the Normal Form Theorem for c.e. predicates.

Note: If $U(e,\bar{x})$ is universal Σ^0_n, then $\neg U(e,\bar{x})$ is universal Π^0_n (and conversely).

Induction step: Let $U(e,y,\bar{x})$ be universal Σ^0_n for $k+1$-ary predicates.

Then $\forall y\ U(e,y,\bar{x})$ is universal Π^0_{n+1} for k-ary predicates since

1. it is in Π^0_{n+1} and
2. for every k-ary Π^0_{n+1}-predicate $P(\bar{x})$ there exists a $k+1$-ary Σ^0_n-predicate $Q(y,\bar{x})$ such that

$$P(\bar{x}) = \forall y\ \underbrace{Q(y,\bar{x})}_{U(e,y,\bar{x})}$$

The arithmetical hierarchy does not collapse

Corollary

For each $n \geq 1$ there exist Σ^0_n-predicates that are not Π^0_n (and conversely).

Proof.

- Let $U_n(e, x)$ be a unary universal Σ^0_n-predicate.
- Then $K_n(x) := U_n(x, x)$ is Σ^0_n.
- Seeking a contradiction, suppose K_n is Π^0_n. Then $\neg K_n$ is Σ^0_n.
- Hence $\neg K_n(x) \equiv U_n(e, x)$ for some e.
- Contradiction for $x = e$.

$\square$