Arithmetical Hierarchy

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DTM vs functions on N

For a partial function f write

- ▶ $f(x) \downarrow \text{ if } x \text{ is in the domain of } f$;
- ▶ $f(x) \uparrow \text{ if } x \text{ is not in the domain of } f$.

 $\varphi_e(x)\colon \mathbb{N} \to_{\rho} \mathbb{N}$ is computed by the DTM with Goedel number e

Facts

- ▶ $A \subseteq \mathbb{N}^k$ is computably enumerable iff A is the domain of some partial recursive function.
- ▶ $A \subseteq \mathbb{N}^k$ is computable iff the characteristic function of A is recursive.
- ► The Diagonal Halting Problem

$$K := \{x \in \mathbb{N} : \varphi_x(x) \downarrow \}$$

is c.e. but not computable.



Properties of recursive functions are not computable

Rice's Theorem

Let C be a class of k-ary recursive functions. Then $\{e \in \mathbb{N} : \varphi_e \in C\}$ is computable iff $C = \emptyset$ or C is the class of all k-ary recursive functions.

Example

None of the following are computable:

- $ightharpoonup K := \{x \in \mathbb{N} : \varphi_x(x) \downarrow \}$
- $ightharpoonup F := \{x \in \mathbb{N} : \varphi_x \text{ has finite domain}\}$
- $ightharpoonup T := \{x \in \mathbb{N} : \varphi_x \text{ is total } \}$

The arithmetical hierarchy of subsets of $\mathbb N$

Idea: Classify problems that are not computable by the complexity of formulas that describe them.

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Example (Diagonal halting problem K)
x \in K \text{ iff } \varphi_x(x) \downarrow \qquad \qquad \sum_{i=1}^n \frac{(\operatorname{config}(x,x,y))_0 = t}{(\operatorname{computable predicate}_{i})_0} 
\text{computable predicate}_{i}
\text{halts after y steps}
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Definition

Let $P(\bar{x})$ be a k-ary predicate on \mathbb{N} , $n \in \mathbb{N}$:

ightharpoonup P is $\sum_{n=0}^{\infty} P$ if there is a computable predicate R:

$$P(\bar{x}) \equiv \underbrace{\exists y_1 \forall y_2 \exists y_3 \dots \exists / \forall y_n}_{n \text{ alternating quantifiers starting with } \exists R(\bar{x}, \bar{y})$$

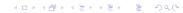
ightharpoonup P is Π_n^0 if there is a computable predicate R:

$$P(\bar{x}) \equiv \underbrace{\forall y_1 \exists y_2 \forall y_3 \dots \exists / \forall y_n}_{n \text{ alternating quantifiers starting with } \forall} : R(\bar{x}, \bar{y})$$

- $ightharpoonup \Sigma_0^0 = \Pi_0^0 = \text{computable predicates}$

Note

The superscript 0 denotes quantification over type-0-objects (elements in \mathbb{N}).



Example

- 1. K is Σ_1^0
- 2. $T = \{e \in \mathbb{N} : \varphi_e \text{ is total}\}$ $e \in T \text{ iff } \forall x \ \varphi_e(x) \downarrow$ $\text{iff } \forall x \ \exists y \ (\text{config}(e, x, y))_0 = t$

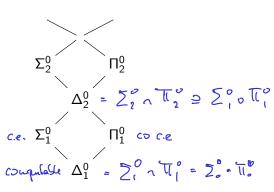
Hence T is Π_2^0 .

3. $F = \{e \in \mathbb{N} : \varphi_e \text{ has finite domain}\}\$ $e \in F \text{ iff } \exists z \forall y \forall x \text{ } (\operatorname{config}(e, x, y))_0 = t \Rightarrow x \leq z$ Hence $F \text{ is } \Sigma_2^0$.

Arithmetical hierarchy

$$\bigcup_{n\in\mathbb{N}}\Sigma_n^0=\bigcup_{n\in\mathbb{N}}\Pi_n^0$$

(sets defined in first order arithmetic, hence called **arithmetical**)



Question

Are all subsets of \mathbb{N} arithmetical?

Closure properties

For proving the previous picture we need some preparation.

Lemma

Let $n \ge 1$.

- 1. Σ_n^0 is closed under existential quantification, Π_n^0 is closed under universal quantification.
- 2. Σ_n^0 , Π_n^0 are both closed under \wedge , \vee , bounded quantifiers $\forall x < y, \exists x < y$, and substitution of total computable functions.
- 3. $\neg \Sigma_n^0 = \Pi_n^0, \neg \Pi_n^0 = \Sigma_n^0$

Proof sketch.

Let R be computable, $P(x,z) = \exists y_1 \forall y_2 \dots R(x,z,y_1,\dots)$ be Σ_n^0 .

1. Claim: $Q(x) := \exists z \ P(x,z) \text{ is } \Sigma_n^0$

$$Q(x) \equiv \exists z \exists y_1 \forall y_2 \dots R(x, z, y_1, y_2, \dots, y_n)$$

$$\equiv \exists u \quad \forall y_2 \dots R(x, (u)_0, (u)_1, y_2 \dots, y_n)$$

Dual argument for \forall and Π_n^0 .

2. **Substitution:** Let f(x) total, computable.

Claim:
$$Q(x) := P(x, f(x))$$
 is Σ_n^0
 $Q(x) \equiv \exists y_1 \forall y_2 \dots \underbrace{R(x, f(x), y_1, y_2, \dots)}_{\text{computable since } R \text{ is}}$

Conjunction: Induct on n (HW).

3. Negation: immediate. de Morgon

Σ_1^0 is computably enumerable

Normal Form Theorem for c.e. sets P is c.e. iff P is Σ_1^0 .

Proof

 \Rightarrow : Let $P \subseteq \mathbb{N}^k$ be c.e.

▶ Then $P = \operatorname{domain} \varphi_e^{(k)}$ for some e (HW).

$$x \in P \text{ iff } \frac{\varphi_e(x) \downarrow}{\exists n \text{ } (\text{config}(e, x, n))_0 = t} \text{ universal } Z^n,$$

$$=: \varphi_{e,n}(x) \downarrow \text{ computes in } n \text{ sleps}$$

▶ P is Σ_1^0 because the predicate $\varphi_{e,n}(x) \downarrow$ (" M_e computes $\varphi_e(x)$ in n steps") is computable.

 \Leftarrow : Let $P(x) \equiv \exists y \ R(x,y)$ for R computable.

- ▶ Then $\psi(x) := \mu y R(x, y)$ is recursive.
- \triangleright $P = \operatorname{domain} \psi$ is c.e.

Dually Π_1^0 is co-c.e.

Universal Σ_n^0 predicates

Idea: Enumerate k-ary Σ_n^0 predicates by a single k+1-ary Σ_n^0 predicate.

Definition

A k+1-ary predicate $U(e,\overline{x})$ is **universal** Σ_n^0 for k-ary predicates if

- 1. $U(e,\bar{x})$ is Σ_n^0 and
- 2. for every k-ary Σ_n^0 predicate $P(\bar{x})$ there is some e such that

$$P(\bar{x}) \equiv U(e,\bar{x}).$$

Universal Π_n^0 -predicates are defined correspondingly.

Example

From the last proof $U(e,x) := \exists n \ \varphi_{e,n}(x) \downarrow \text{ is universal } \Sigma_1^0$.



Enumeration Theorem

For all $k, n \ge 1$, universal Σ_n^0 - and Π_n^0 -predicates exist.

Proof by induction on n and k:

Base case: $U(e,x) := \exists m \ \varphi_{e,m}(x) \downarrow \text{ is universal } \Sigma_1^0 \text{ by the Normal Form Theorem for c.e. predicates.}$

Note: If $U(e, \bar{x})$ is universal Σ_n^0 , then $\neg U(e, \bar{x})$ is universal Π_n^0 (and conversely).

Induction step: Let $U(e, y, \bar{x})$ be universal Σ_n^0 for k+1-ary predicates.

Then $\forall y \ U(e, y, \bar{x})$ is universal Π_{n+1}^0 for k-ary predicates since

- 1. it is in Π_{n+1}^0 and
- 2. for every k-ary Π_{n+1}^0 -predicate $P(\bar{x})$ there exists a k+1-ary Σ_n^0 -predicate $Q(y,\bar{x})$ such that

$$P(\bar{x}) = \forall y \ \underbrace{Q(y,\bar{x})}_{(x_1,y_1)} \in \mathcal{A}$$



The arithmetical hierarchy does not collapse

Corollary

For each $n \ge 1$ there exist Σ_n^0 -predicates that are not Π_n^0 (and conversely).

Proof.

- ▶ Let $U_n(e,x)$ be a unary universal $\sum_{n=0}^{\infty}$ -predicate.
- ▶ Then $K_n(x) := U_n(x,x)$ is Σ_n^0 .
- ▶ Seeking a contradiction, suppose K_n is Π_n^0 . Then $\neg K_{\kappa}$ is Σ_n^0 .
- ► Hence $\neg K_n(x) \equiv U_n(e,x)$ for some e.
- ▶ Contradiction for x = e.

