Normal Form Theorem for recursive functions

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Gödel numbering of DTM

To show that computable functions on $\mathbb{N}$ are recursive we first encode DTMs and their configurations as natural numbers.

- Let $2 = p_0 < p_1 < p_2 < \dots$ be the enumeration of all primes.
- $\forall x \in \mathbb{N} \setminus \{0\}$ $\exists$ unique $(x)_0, (x)_1, \dots \in \mathbb{N}$, almost all 0:

  $$x = \prod_{i \in \mathbb{N}} p_i^{(x)_i}$$

  Prime Factorization Thm

  The exponent $(x)_i$ of p_i is primitive recursive in x, i.
  - Recall $i \mapsto p_i$ is primitive recursive,
  - $(x)_i = \mu(t < x)\,[p_i^{t+1}$ does not divide $x]$ is primitive recursive.
- Encode a tuple $(a_0, \dots, a_n) \in \mathbb{N}^*$ as

  $$a := p_0^{a_0+1} \cdots p_n^{a_n+1} \in \mathbb{N}.$$

  Then n and a_i for all $i \leq n$ are primitive recursive in a.

Definition

Let $M = (Q, \{0, 1\}, \Gamma, s, t, r, \delta)$ be a DTM with

- n states $Q = \{1, \dots, n\}$,
- tape alphabet $\Gamma = \{\gamma_0 = \sqcup, \gamma_1 = 1, \gamma_2 = 0, \gamma_4, \dots, \gamma_m\}$,

We define

$$a_0 := 2^{|Q|}3^{|\Gamma|}5^s 7^t 11^r$$

and encode the i-th transition $\delta(q, \gamma_k) = (p, \gamma_\ell, d)$ for $1 \le i \le (n-2)m$ as

$$a_i := 2^q 3^k 5^p 7^\ell 11^{1+d}.$$

Then the **prime power encoding** of M is

$$\sharp(M) = \prod_{i=0}^{(n-2)m} p_i^{a_i}.$$

Definition

A configuration (q, α, k) of the DTM M is encoded as

$$\sharp(q, \alpha, k) := 2^q 3^{\prod_{i \in \mathbb{N}} p_i^{\alpha(i)}} 5^k$$

Note

- The encoding $\sharp(M)$ essentially IS (the transition function of) M.
- All constituents of $\sharp(M)$ and of the encoding of a configuration are primitive recursive.
- The predicate "x is the prime power encoding of a DTM" is primitive recursive.
- Let $c_0 < c_1 < \ldots$ be the enumeration of prime power encodings $\sharp(M)$ of DTMs.
  $\mathbb{N} \to \mathbb{N}$, $e \mapsto c_e$, is a primitive recursive enumeration of all DTMs.

$c_0 := \ldots$

$c_{e+1} := \mu t \; [\, t > c_e \;\&\; t \text{ is DTM-code} \,]$

bounded by code for DTM with more states than M_e has

Computable functions are recursive

Recall our encoding of tuples of natural numbers as strings:

- $n \in \mathbb{N}$ is represented as $\underbrace{1\ldots1}_{n+1}$
- (m, n) as $1^{m+1}01^{n+1}$

Definition

For $e \in \mathbb{N}$, let $\varphi_e^{(k)} : \mathbb{N}^k \to_p \mathbb{N}$ **be the k-ary partial function computed by the e-th DTM** M_e [i.e. $\sharp(M_e) = c_e$].

$\varphi_e^{(2)}(1,2) = 3$ iff M_e on input 110111 halts with tape content 1111

$\varphi_e^{(1)} \neq \varphi_e^{(2)}$ but computed by the same DTM M_e

Normal Form Theorem (Kleene)

For every $k \in \mathbb{N} \setminus \{0\}$ there exists a primitive recursive predicate $T^{(k)}(e, \bar{x}, y)$ and a primitive recursive function u such that

$$\varphi_e^{(k)}(\bar{x}) = u\big(\mu y\, T^{(k)}(e, \bar{x}, y)\big).$$

Proof sketch for $k = 1$.

Claim 1:

$$\mathrm{succ}(c) := \sharp(\text{successor of the configuration with encoding } c)$$

is primitive recursive.

Let c be the encoding of a configuration. Then

- $q := (c)_0$ is its state,
- $k := (c)_2$ is the position of the head,
- $i := ((c)_1)_k$ is the index of the cell content γ_i.

Then

$$\mathrm{succ}(e, c) := \begin{cases} 2^p 3^{(c)_1 \cdot p_k^{-i+j}} 5^{\max(k+d,0)} & \text{if } \delta(q, \gamma_i) = (p, \gamma_j, d), \\ c & \text{if } q \in \{t, r\}. \end{cases}$$

Note that p, j, d are primitive recursive from e, q, i. Claim 1 is proved.

Claim 2:

$$\mathrm{config}(e, x, n) := \sharp(\text{configuration of } M_e \text{ on input } x \text{ at step } n)$$

is primitive recursive by the recursion scheme:

$$\mathrm{config}(e, x, 0) := 2^s 3^{\prod_{i \leq |x|} p_i^{x_i}} 5^0$$
$$\mathrm{config}(e, x, n+1) := \mathrm{succ}(\mathrm{config}(e, x, n))$$

Claim 3:
Assume M_e has computed $\varphi_e(x)$ iff it is in the accept state t. Then

$$\mathrm{halt}(e, x) := \mu n \, [(\mathrm{config}(e, x, n))_0 = t]$$

is recursive and yields the number of steps n for M_e to write $\varphi_e(x)$ on the tape and halt.

Then $\varphi_e(x)$ is primitive recursive from $\mathrm{config}(e, x, \mathrm{halt}(e, x))$. $\square$

Consequences of the Normal Form Theorem

- A partial function on $\mathbb{N}$ is computable (by a DTM) iff it is recursive (supporting the Church-Turing thesis again).
- Kleene's T-predicate and the function u in the Normal Form Theorem are universal, i.e., independent of the function φ_e.
- Recursive functions can be defined with at most one application of unbounded search μ.

Corollary (Enumeration Theorem and Universal Machine)

$\psi(e, \bar{x}) := \varphi_e^{(k)}(\bar{x})$ is recursive.

Proof.

Immediate from the Normal Form Theorem. □

Question

Why does the diagonalization argument not yield a computable function $d(x) := \varphi_x(x) + 1$ that is not recursive?

Dually to the Enumeration Theorem we have:

Parameter Theorem/S^m_n-Theorem

For all $m, n \geq 1$ there exists a primitive recursive function S^m_n such that $\forall x \in \mathbb{N}, \bar{y} \in \mathbb{N}^m, \bar{z} \in \mathbb{N}^n$:

$$\varphi_{S^m_n(x,\bar{y})}(\bar{z}) = \varphi_x(\bar{y}, \bar{z})$$

Note

$\bar{y}$ can be regarded as fixed parameter for the DTM $M_{S^m_n(x,\bar{y})}$ and the index $S^m_n(x, \bar{y})$ is computable.

Proof.

- Let M be the DTM that on input $\bar{z}$ simulates the DTM M_x on input $(\bar{y}, \bar{z})$.
- Then $\cancel{\sharp(M)} =: S^m_n(x, \bar{y})$ is primitive recursive from $x, \bar{y}$.

the number of M
in our enumeration of DTMs

$\square$