Recursive functions

Peter Mayr

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Definition (Formulation by Kleene 1936)

The class of (μ-)**recursive** functions is the least class of finitary partial functions on $\mathbb{N}$ that contains

primitive recursive

1. contains 0, successor, all projections,
2. is closed under composition,
3. is closed under primitive recursion,
4. is closed under search (minimization) μ:
   if $f : \mathbb{N}^{k+1} \to \mathbb{N}$ is recursive, then

$$g : \mathbb{N}^k \to \mathbb{N},$$
$$x \mapsto (\mu y)\,[f(x, y) = 1 \ \&\ (x, \cancel{y}\,t) \in \mathrm{dom} f \ \forall t \leq y]$$

$\min\{y \in \mathbb{N} \mid f(x,y) = 1 \ \&\ (x,t) \in \mathrm{dom} f \ \forall t \leq y\}$

is recursive. (Here $g(x)$ is undefined if no such y exists.)

Theorem (Turing 1936)

Every recursive function is computable.

Proof sketch..

Show that computable functions contain 0, successor, projections and are closed under composition, primitive recursion and search.

1) composition clear

2) primitive recursion: $f(x,0) := g(x)$

$f(x,y+1) := h(x,y,f(x,y))$

with g, h computed by DTM M_g, M_h

Compute $f(x,0), \ldots, f(x,y)$ to get $f(x,y+1)$.

3) search μ: Assume f is computed by M_f.

Compute $f(x,0), f(x,1), \ldots$ until you find $f(x,y)=1$.

May not halt!

$\square$