

Ackermann function

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Question

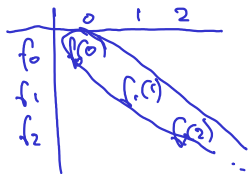
Primitive recursive functions are computable.

Is every computable total function primitive recursive?

Answer 1

No, by diagonalization there is no computable enumeration of all computable total functions on \mathbb{N} .

- Let
- 1) $(f_n)_{n \in \mathbb{N}}$ be a list of all computable functions $\mathbb{N} \rightarrow \mathbb{N}$
 - 2) $g(n, m)$ computable such that $g(n, m) = f_n(m) \quad \forall n, m \in \mathbb{N}$
 - 3) f_n total for all $n \in \mathbb{N}$.



$f_n(n) + 1 = g(n, n) + 1$ is computable
but not in the list $(f_n)_{n \in \mathbb{N}}$

Note: primitive recursive functions are c.e. by their definitions,
hence not all computable functions.

Answer 2

Some computable functions grow too fast to be primitive recursive.

Knuth's up arrow notation.

$a \uparrow^n b$ is defined by $a \uparrow b := \underbrace{a \cdots a}_b$

$$a \uparrow\uparrow b := \underbrace{a \overset{\cdot \cdot \cdot a}{\uparrow}}_b$$

$$a \uparrow^{n+1} b := \underbrace{a \uparrow^n (a \uparrow^n \dots a)}_b$$

Definition

For $m, n \in \mathbb{N}$ define the **Ackermann function** $A(m, n)$ by

$$A(0, n) := n + 1$$

$$A(m + 1, 0) := A(m, 1)$$

$$A(m + 1, n + 1) := A(m, A(m + 1, n))$$

(Not a primitive recursion scheme as it uses recursion over itself.)

Example

$$A(1, n) = n + 2$$

$$A(2, n) = 2n + 3$$

$$A(3, n) = 2^{n+3} - 3$$

$$A(4, n) = \underbrace{2^{2^{\cdot^{\cdot^2}}}}_{n+3} - 3$$

$$A(5, n) = 2 \uparrow \uparrow \uparrow (n + 3) - 3$$

Facts

1. $A(m, n)$ is a total, computable function.
2. A is strictly increasing in each argument.
3. $A(m, n + 1) \leq A(m + 1, n)$
4. $A(\ell, A(m, n)) < A(\ell + m + 2, n)$

Proof ideas

1. Induction on (m, n) in lex order.
2. Induction on m, n respectively.
3. Induction on n .
4. $A(\ell, A(m, n)) < A(\ell + m, A(\ell + m + 1, n)) \stackrel{\text{Def.}}{=} A(\ell + m + 1, n + 1) \stackrel{3}{\leq} A(\ell + m + 2, n)$.

Majorization Lemma

For every primitive recursive $f(\bar{x})$ there exists $M \in \mathbb{N}$ such that

$$\forall \bar{x}: f(\bar{x}) < A(M, \max(\bar{x})).$$

Proof by induction on the representation of f .

Base cases $f = 0, s, p_i^k$ are straightforward for $M = 0, 1$.

Induction step:

1) **Composition:** Let $f(\bar{x}) := g(h_1(\bar{x}), \dots, h_n(\bar{x}))$ for g, h_1, \dots, h_n primitive recursive.

Let $x := \max(\bar{x})$. By induction assumption we have $G, H \in \mathbb{N}$ such that

$$g(\bar{y}) < A(G, \max(\bar{y})), \quad h_i(\bar{x}) < A(H, x) \text{ for all } i.$$

Then

$$f(\bar{x}) < A(G, \max(h_i(\bar{x}))) < A(G, A(H, x)) < A(G + H + 2, x).$$

2) **Recursion scheme:** Let $f(\bar{x}, y)$ be defined by

$$f(\bar{x}, 0) := g(\bar{x})$$

$$f(\bar{x}, y + 1) := h(\bar{x}, y, f(\bar{x}, y))$$

for g, h primitive recursive.

By induction assumption we have $G, H \in \mathbb{N}$ such that

$$g(\bar{x}) < A(G, x), \quad h(\bar{x}, y, z) < A(H, \max(x, y, z)).$$

Claim: $f(\bar{x}, y) < A(F, x + y)$ for $F := \max(G, H) + 1$ (†)

Induct on y :

Base case:

$$f(\bar{x}, 0) = g(\bar{x}) < A(G, x) < A(F, x)$$

Induction step:

$$f(\bar{x}, y + 1) = h(\bar{x}, y, f(\bar{x}, y)) < A(H, \max(x, y, f(\bar{x}, y)))$$

By the induction hypothesis and $x, y < A(F, x + y)$,

$$\max(x, y, f(\bar{x}, y)) < A(F, x + y).$$

Now Claim (†) follows from

$$f(\bar{x}, y+1) < A(H, A(F, x+y)) \stackrel{Df.}{\leq} A(F-1, A(F, x+y)) = A(F, x+y+1).$$

Finally let $z := \max(x, y)$. Using Claim (\dagger)

$$f(\bar{x}, y) < A(F, 2z) < A(F, 2z + 3) = A(F, A(2, z)) < A(F + 4, z).$$

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The Majorization Lemma is proved. □

Corollary

The Ackermann function $A(m, n)$ is not primitive recursive.

Proof.

Seeking a contradiction, suppose otherwise.

- ▶ Then $f(n) := A(n, n)$ is primitive recursive.
- ▶ By the Majorization Lemma we have $M \in \mathbb{N}$ such that $f(n) < A(M, n)$.
- ▶ Then $f(M) < A(M, M)$ is a contradiction.

