# Application: word problems

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# Rewriting systems

Book, Otto. String-rewriting Systems. 1993.

## Example

(S, ·) with associative

Presentation of a monoid (semigroup with 1):

$$\langle \underbrace{a, b}_{\text{generators}} \ : \ \underbrace{ab \overset{\rightarrow}{=} 1, ba \overset{\rightarrow}{=} 1}_{\text{relations}} \rangle$$

Ex. $a\underbrace{ba}_{\to 1}a \to a1a = aa$

$\underbrace{ab}_{\to 1}aa \to 1aa = aa$

## Definition

- A **string rewriting system (SRS)** $R$ over a finite alphabet $\Sigma$ is a subset of $\Sigma^* \times \Sigma^*$ (rewriting rules).
- For $u, v \in \Sigma^*$
$$u \to_R v$$
if $\exists(\ell, r) \in R \; \exists x, y \in \Sigma^* : \; u = x\ell y, v = xry.$
- $\overset{*}{\leftrightarrow}_R$ is the reflexive, transitive, symmetric closure of $\to_R$. Then $\overset{*}{\leftrightarrow}_R$ is a congruence on the free monoid $(\Sigma^*, \cdot)$.
- $M_R := \Sigma^* / \overset{*}{\leftrightarrow}_R$ is the monoid **presented by** $\langle \Sigma \; : \; R \rangle$.

# Word problem for semigroups

### Word problem for SRS $R$ on $\Sigma$

**Input:** $u, v \in \Sigma^*$
**Question:** Is $u \overset{*}{\leftrightarrow}_R v$?

### Theorem (Post 1947)

There exist a finite SRS with undecidable word problem (c.e. but not computable).

### Proof idea

Encode DTM as SRS in the following.

# DTM as SRS

Let $M = (Q, \Sigma, \Gamma, s, t, r, \delta)$ be a DTM with bi-infinite tape.

Consider a configuration $(q, \; \ldots \llcorner a_\ell \ldots a_r \llcorner \ldots, \; n)$ as string

state    tape content    position

$$h a_\ell \ldots a_{n-1} q a_n \ldots a_r h$$

delimiter

over $\Omega := Q \,\dot\cup\, \Gamma \,\dot\cup\, \{h, t_1, t_2\}$.

"copies of accept state"

Define SRS $S(M)$. For $a, a', b \in \Gamma, q, q' \in Q$ let

encode transitions

1. $qa \to a'q'$     if $\delta(q, a) = (q', a', +1)$
2. $qh \to a'q'h$     if $\delta(q, \llcorner) = (q', a', +1)$
3. $bqa \to q'ba'$     if $\delta(q, a) = (q', a', -1)$
4. $hqa \to hq'\llcorner a'$     if $\delta(q, a) = (q', a', -1)$

delete tape after acceptance

5. $t \to t_1$
6. $t_1 a \to t_1$
7. $a t_1 h \to t_1 h$
8. $h t_1 h \to t_2$

# Rewriting configurations

## Lemma

For $u, v, u', v' \in \Gamma^*$ and $q, q' \in Q$ TFAE:

1. $(q,\ \sqcup uv\sqcup,\ \text{position of } v_1) \vdash_M^* (q',\ \sqcup u'v'\sqcup,\ \text{position of } v'_1)$
2. $\exists m, n \in \mathbb{N}:\ huqvh\ \overset{*}{\rightarrow}_{S(M)}\ h\sqcup^m u'q'v'\sqcup^n h$

## Proof.

1. $\Rightarrow$ 2. is clear by definition of the rewriting rules 1-4.
2. $\Rightarrow$ 1. follows since in item 2. only rules 1-4 are applied as no $t_1, t_2$ are introduced. □

## Corollary

Let $x \in \Sigma^*$. Then $hsxh\ \overset{*}{\rightarrow}_{S(M)}\ t_2$ iff $x \in L(M)$.

## Proof.

$t_2$ can only be introduced from an accepting configuration via rules 5-8. <u>Almost</u> done but $\overset{*}{\leftrightarrow}$ is not just one way. □

# Reducing equivalence to rewriting

## Lemma

Let $w \in \Omega^*$. Then $w \overset{*}{\leftrightarrow}_{S(M)} t_2$ iff $w \overset{*}{\rightarrow}_{S(M)} t_2$.

## Proof.

$\Rightarrow$: Assume $w \overset{*}{\leftrightarrow}_{S(M)} t_2$.

- Either $w = t_2$ or $w = huqvh$ for some $u, v \in \Gamma^*, q \in Q \cup \{t_1\}$ since no rule changes the number of "states" $Q \cup \{t_1, t_2\}$.
- Consider a **shortest path** connecting $w \neq t_2$ and $t_2$ via the symmetric closure $\leftrightarrow = \leftarrow \cup \rightarrow$:

$$w = huqvh = w_0 \leftrightarrow w_1 \leftrightarrow \cdots \leftrightarrow w_k = t_2$$

- Then $w_{k-1} = ht_1h \;\rightarrow\; t_2 = w_k$.
- Let $\ell \in \mathbb{N}$ minimal such that $w_\ell$ contains $t_1$. Then

$$w_{\ell-1} = hu_{\ell-1}tv_{\ell-1}h \;\rightarrow\; hu_{\ell-1}t_1v_{\ell-1}h = w_\ell.$$

- Clearly $w_{\ell-1} \overset{*}{\rightarrow} t_2$.

- It remains to show $w \xrightarrow{*} w_{\ell-1}$.
- Note that $w_{\ell-2} \rightarrow w_{\ell-1}$ since $M$ stops when reaching $t$.
- Let $m \in \mathbb{N}$ maximal such that

$$w \xrightarrow{*} w_{m-1} \leftarrow w_m \rightarrow w_{m+1} \xrightarrow{*} w_{\ell-1} \rightarrow w_\ell$$

- Then $w_{m-1} = w_{m+1}$ represents the unique successor configuration of $w_m$.
- We can skip $w_m$ above to get a shorter path from $w$ to $t_2$.
- Hence our minimal path from $w$ to $t_2$ cannot contain any $\leftarrow$. Thus $w \xrightarrow{*} t_2$. $\square$

# SRS are equivalent to DTM

### Corollary

Let $x \in \Sigma^*$. Then $hsxh \overset{*}{\leftrightarrow}_{S(M)} t_2$ iff $x \in L(M)$.

### Note

- The language of any DTM many-one reduces to the word problem of the corresponding SRS.
- Conversely word problems can clearly be solved by NTM.
- SRS are a **Turing complete** model of computation (exactly as powerful as DTM, $\lambda$-calculus, ...).

# Word problem for semigroups is undecidable

For a DTM with not computable language (e.g. AP), the corresponding SRS is not computable either. We proved:

### Theorem (Post 1947)

There exist a finite SRS with undecidable word problem (c.e. but not computable).

### Note

- Non-trivial properties of finite SRS are undecidable (Rice's Theorem).
- Undecidability of the word problem for groups follows with similar ideas but much harder details (Novikov 1955).
- 1-relator groups have decidable word problem (Magnus 1932).
- Matiyasevich (1967) gave an undecidable SRS with 2 generators and 3 relations.
- Open: Are 1-relator SRS decidable?
  1-relator inverse monoids have undecidable word problem