Deterministic and nondeterministic automata

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Recall

Definition

A **deterministic finite automaton (DFA)** is a 5-tuple $M = (Q, \Sigma, \delta, s, F)$ with

- Q a finite set (**states**),
- Σ a finite set (**input alphabet**),
- $\delta \colon Q \times \Sigma \to Q$ the **transition function**,
- $s \in Q$ the **start state**,
- $F \subseteq Q$ the set of **final/accepting states**.

M **accepts** $w \in \Sigma^*$ if $\delta^*(s, w) \in F$ for the extension δ^* of δ to Σ^*.

$$L(M) := \{w \in \Sigma^* \ : \ M \text{ accepts } w\}$$

is the **language of** M.

Example

Is there a DFA M_3 such that
$L(M_3) = \{w \in \{0,1\}^* \ : \ 001 \text{ is a substring of } w\}$?

Idea: scan w for 001

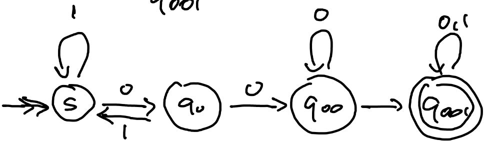
States: start s ... no part of 001 seen yet

q_0 -- just saw 0

q_{00} just saw 00

q_{001} 001

(states represent finite memory of DFA)



Nondeterministic finite automata

- **Deterministic:** current state and input symbol uniquely determine next state
- **Nondeterministic:** several choices for next state
  - Interpretation: all possible transitions are done in parallel/the 'right' one is guessed.
  - Not a realistic model of computation but a useful theoretical device for its analysis.

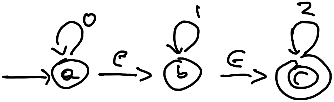
Definition

A **nondeterministic finite automaton with ϵ-transitions (ϵ-NFA)** is a 5-tuple $(Q, \Sigma, \Delta, s, F)$ like a DFA except that

$$\Delta \colon Q \times \left(\Sigma \cup \{\epsilon\}\right) \to P(Q) \qquad (P(Q) \ldots \text{power set of } Q)$$

- Recall ϵ is the empty word, not an element in Σ.
- ϵ-transitions allow the NFA to change from a state q to any state in $\Delta(q, \epsilon)$ without input.
- Wlog, the sets $T := \Delta(q, a)$ for any $a \in \Sigma \cup \{\epsilon\}$ are ϵ-**closed** (i.e. if $t \in T$, then also $\Delta(t, \epsilon) \subseteq T$).

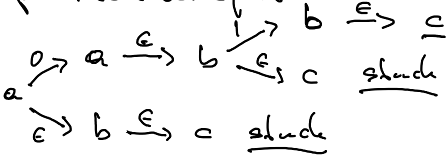
Example ϵ-NFA N



$$\Delta(a,1) = \emptyset$$

$$\Delta(a,\epsilon) = \{a, b, c\}$$

Computation on input 01



branch processes the whole input & reaches final state c
accepting branch

$$L(N) = \{ 0^m 1^n 2^o \mid m, n, o \in \mathbb{N} \}.$$

Definition

For an ϵ-NFA $N = (Q, \Sigma, \Delta, s, F)$ the **extended transition function**

$$\Delta^*\colon Q \times \Sigma^* \to P(Q)$$

is defined inductively for $q \in Q, w \in \Sigma^*, a \in \Sigma$ by

$$\begin{aligned} \Delta^*(q,\epsilon) &:= \Delta(q,\epsilon) \\ \Delta^*(q,wa) &:= \textstyle\bigcup_{r \in \Delta^*(q,w)} \Delta(r,a) \end{aligned}$$

assuming all $\Delta(q,\epsilon)$ and $\Delta(r,a)$ are ϵ-closed.

$\Delta^* |_{Q \times (\Sigma \cup \{\epsilon\})} = \Delta$

- N **accepts** w if $\Delta^*(s,w) \cap F \neq \emptyset$
  (i.e. N accepts w iff $\exists$ some path from s to a state in F that is labelled by w).
- N **rejects** w otherwise.

The **language of** N is

$$L(N) := \{w \in \Sigma^* \ : \ M \text{ accepts } w\}.$$

Note

- Every DFA can be considered as ϵ-NFA with $\Delta(q,a) := \{\delta(q,a)\}$ (singleton) and $\Delta(q,\epsilon) := \emptyset$.
- Every language accepted by a DFA is also accepted by some ϵ-NFA. What about the converse?

Theorem (Subset construction (Rabin, Scott 1959))

Let $N = (Q, \Sigma, \Delta, s, F)$ be an ϵ-NFA with all $\Delta(q,a)$ ϵ-closed. Let $M = (Q', \Sigma, \delta, s', F')$ be the DFA with

- $Q' := P(Q)$,
- $\delta(R,a) := \bigcup_{q \in R} \Delta(q,a)$ for $R \subseteq Q, a \in \Sigma$,
- $s' := \Delta(s,\epsilon)$,
- $F' := \{R \in P(Q) \ : \ R \cap F \neq \emptyset\}$.

Then $L(N) = L(M)$.

Proof

First show for all $w \in \Sigma^*$ that

$$\delta^*(s', w) = \Delta^*(s, w) \qquad (\dagger)$$

by induction on $|w|$.

Base case: For $w = \epsilon$,

$$\delta^*(s', \epsilon) = s' = \Delta(s, \epsilon) = \Delta^*(s, \epsilon).$$

Induction hypothesis: $(\dagger)$ holds for $w \in \Sigma^*$ of length n.
Let $a \in \Sigma$. Then

$$\begin{aligned}
\delta^*(s', wa) &= \delta\big(\,\delta^*(s', w), a\big) && \text{by definition of } \delta^* \\
&= \delta\big(\,\Delta^*(s, w), a\big) && \text{by induction hypothesis} \\
&= \textstyle\bigcup_{q \in \Delta^*(s,w)} \Delta(q, a) && \text{by definition of } \delta \\
&= \Delta^*(s, wa) && \text{by definition of } \Delta^*
\end{aligned}$$

Hence $(\dagger)$ is proved.

Proof, continued

Note: N accepts w iff $\Delta^*(s, w) \cap F \neq \emptyset$
iff $\delta^*(s', w) \in F'$ by (†) and the definition of F'
iff M accepts w. □

Note

The subset construction translates an NFA with $|Q|$ states into a DFA with $2^{|Q|}$ states. Often fewer suffice.

Example, continued

Recall t ϵ-NFA N with 3 states, $L(N) = \{0^\ell 1^m 2^n \ : \ \ell, m, n \in \mathbb{N}\}$.
There is a DFA M with $L(M) = L(N)$ and

δ	0	1	2
$\{a, b, c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{c\}$
$\{b, c\}$	$\emptyset$	$\{b, c\}$	$\{c\}$
$\{c\}$	$\emptyset$	$\emptyset$	$\{c\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Since the other subsets cannot be reached from the starting state $\{a, b, c\}$, they can be omitted.