38. Strongly minimal formulas

Strong minimality

Let $\mathcal{M}$ be an $\mathcal{L}$-structure, let $\phi(\bar{x})$ an $\mathcal{L}_M$-formula defining an infinite set $\phi(\mathcal{M})$.

- $\phi(\mathcal{M})$ is **minimal in $\mathcal{M}$** if any definable $D \subseteq \phi(\mathcal{M})$ is finite or cofinite in $\phi(\mathcal{M})$; then we also call ϕ minimal.
- $\phi(\mathcal{M})$ and ϕ are **strongly minimal** if ϕ is minimal in every elementary extension of $\mathcal{M}$.
- A theory T is **strongly minimal** if $x = x$ is strongly minimal, i.e., the universe of every model of T is strongly minimal.

Example

The following theories are strongly minimal (using quantifier elimination):

1. Infinite sets.
2. Infinite $\mathcal{F}$-vector spaces over a field $\mathcal{F}$.
3. ACF_p.

Algebraic formulas

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq D \subseteq M$.

- An $\mathcal{L}_A$-formula $\phi(x)$ is **algebraic** if $\phi(\mathcal{M})$ is finite.
- $a \in M$ is **algebraic over** A if it realizes an algebraic $\mathcal{L}_A$-formula.
- $\mathrm{acl}_D(A) := \{a \in D \mid a \text{ is algebraic over } A\}$ is the **algebraic closure** of A (in D).

Example

In ACF, a is algebraic over A iff a is a root of a polynomial with coefficients in the field generated by A.

acl_D is a closure operator and satisfies the

Exchange Principle (Marker, 6.1.4)

Let $D \subseteq M$ be strongly minimal, $A \subseteq D$, $a, b \in D$. Then

$$a \in \mathrm{acl}_D(A \cup \{b\}) \setminus \mathrm{acl}_D(A) \;\Rightarrow\; b \in \mathrm{acl}_D(A \cup \{a\}).$$

Dimension

We can generalize linear independence in vector spaces to strongly minimal sets D.

- $A \subseteq D$ is **independent** if $a \notin \mathrm{acl}(A \setminus \{a\})$ for any $a \in A$.
- A is a **basis** for $Y \subseteq D$ if $A \subseteq Y$ is independent and $\mathrm{acl}_D(A) = \mathrm{acl}_D(Y)$.
- Bases for any $Y \subseteq D$ exist and they all have the same cardinality, called the **dimension** of Y.

Note that if $\mathcal{L}$ is countable and D uncountable, then $\dim(D) = |D|$.

Theorem (Marker 6.1.11)

Let $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2 \models T$ with $\mathcal{M}_0 \prec \mathcal{M}_1, \mathcal{M}_2$ and $A \subseteq M_0$, let ϕ be a strongly minimal $\mathcal{L}_A$-formula. If $\dim \phi(\mathcal{M}_1) = \dim \phi(\mathcal{M}_2)$, then there exists a bijective partial elementary map $f\colon \phi(\mathcal{M}_1) \to \phi(\mathcal{M}_2)$.

[Proof idea: Bases form (totally) indiscernibles.]

Existence of strongly minimal formulas

Lemma (Marker, 6.1.13)

Every model of an ω-stable theory has a minimal formula.

[Contrapositive proof by constructing a binary tree of formulas.]

Corollary (Marker, 6.1.15)

If T has no Vaughtian pairs, then every minimal formula is strongly minimal.

The Baldwin-Lachlan Theorem

Theorem (Baldwin-Lachlan)

Let κ be uncountable. A countable theory T is κ-categorical iff T is ω-stable and has no Vaughtian pairs.

Proof sketch.

$\Rightarrow$ We showed ω-stability and discussed Vaughtian pairs (Slides 37).

$\Leftarrow$ Assume T is ω-stable and has no Vaughtian pairs.
Then T has a prime model $\mathcal{M}_0$ (Slides 34) and a strongly minimal formula with parameters from M_0 (above).
Let $\mathcal{M}_1, \mathcal{M}_2 \models T$ have cardinality κ.
Then $\mathcal{M}_0 \prec \mathcal{M}_1, \mathcal{M}_2$ and $\dim \phi(\mathcal{M}_1) = \dim \phi(\mathcal{M}_2) = \kappa$.
By Theorem 6.6, there is a partial elementary bijection $f\colon \phi(\mathcal{M}_1) \to \phi(\mathcal{M}_2)$.
Since $\mathcal{M}_1$ is a prime extension of $\phi(\mathcal{M}_1)$ and $\mathcal{M}_2$ is a minimal extension of $\phi(\mathcal{M}_2)$ (Marker, Lemma 6.1.17), f extends to an isomorphism $\mathcal{M}_1 \to \mathcal{M}_2$. ☐

Since the Baldwin-Lachlan characterization of uncountably categorical theories does not depend on κ, it implies:

Morley's Categoricity Theorem

If a complete countable theory is κ-categorical for some uncountable κ, then T is κ-categorical for every uncountable κ.

Their main goal was actually to show the following.

Theorem (Baldwin-Lachlan)

If T is uncountably categorical but not $\aleph_0$-categorical, then $I(T, \aleph_0) = \aleph_0$, i.e. T has countably many countable models up to isomorphism

Current research in and applications of model theory

- Classification
  - NIP, NSOP (`https://forkinganddividing.com/`)
  - abstract elementary classes (Shelah's eventual categoricity conjecture)
- Combinatorics
  - approximate subgroups (Breuillard–Green–Tao)
  - structural Ramsey Theory
  - Constraint Satisfaction Problems
- Geometry
  - o-minimality (e.g. RCF)
  - diophantine geometry (André-Oort Conjecture)
- Differential algebras