37. Uncountable categoricity implies ω-stability and no Vaughtian pairs

κ-categorical for $\kappa > \aleph_0$ implies ω-stable

Theorem (Tent-Ziegler, 5.2.4)

A countable theory T which is categorical for some $\kappa > \aleph_0$ is ω-stable.

Proof (by contraposition).

Let $\mathcal{N} \models T$, $A \subseteq N$ countable and $S_1^{\mathcal{N}}(A)$ uncountable.
Let $(b_i)_{i \in I}$ be $\aleph_1$ many elements in N with distinct types over A.
Let $\mathcal{M}_0 \prec \mathcal{N}$ of cardinality $\aleph_1$ containing A and $(b_i)_i \in I$.
Then we have $\mathcal{M} \succ \mathcal{M}_0$ of cardinality κ realizing uncountably many types over countable A.
By a previous Corollary (TZ, 5.1.9) T also has a model of cardinality κ realizing only countably many types over A.
Hence T is not κ-categorical. □

A characterization of κ-categoricity

Theorem (Tent-Ziegler, 5.2.11)

A countable theory T is κ-categorical iff all models of cardinality κ are saturated.

Proof.

$\Leftarrow$ follows from the uniqueness of κ-categorical structures.

$\Rightarrow$ For $\kappa = \aleph_0$ this is part of Ryll-Nardzewski.
For $\kappa > \aleph_0$, categorical implies ω-stable by the previous Theorem, hence κ-stable by (TZ, 5.2.6).
By Lemma (TZ, 5.2.9), all models of T of cardinality κ are saturated. $\square$

Vaughtian pairs

Fact: The size of a definable set in a model of an uncountably categorical theory depends on the size of the model.

For an $\mathcal{L}$-formula $\phi(\bar{x})$ and $\mathcal{L}$-structure $\mathcal{M}$ let

$$\phi(\mathcal{M}) := \{\bar{x} \in M^n \mid \mathcal{M} \models \phi(\bar{x})\}$$

be the set defined by ϕ in $\mathcal{M}$.

For $\kappa > \lambda \geq \aleph_0$, a theory T has a (κ, λ)-**model** if there exists $\mathcal{M} \models T$ with $|M| = \kappa$ and a formula ϕ with $|\phi(\mathcal{M})| = \lambda$.

Lemma

(κ, λ)-models prevent κ-categoricity.

Proof.

Let $\mathcal{M}$ be a (κ, λ)-model of T with $|\phi(\mathcal{M})| = \lambda$. Add κ new constants to the language and extend T by sentences that ϕ holds on all these constants. Since $\lambda \geq \aleph_0$, the extended theory is satisfiable by Compactness, hence has a model $\mathcal{N}$ of cardinality κ with $|\phi(\mathcal{N})| = \kappa$. Then $\mathcal{M} \not\cong \mathcal{N}$ and T is not κ-categorical. □

Vaught's Two-cardinal Theorem

If T has a (κ, λ)-model for $\kappa > \lambda \geq \aleph_0$, then T has an $(\aleph_1, \aleph_0)$-model.

Proof.

Marker 4.3.34. □

T has a **Vaughtian pair** of models $\mathcal{M} \prec \mathcal{N}$ if there exists an $\mathcal{L}_M$-formula ϕ such that

- $\mathcal{M} \neq \mathcal{N}$,
- $\phi(\mathcal{M})$ is infinite,
- $\phi(\mathcal{M}) = \phi(\mathcal{N})$.

Lemma

If T has a (κ, λ)-model for $\kappa > \lambda \geq \aleph_0$, then T has a Vaughtian pair.

The Baldwin-Lachlan Theorem

We now have the main concepts needed for the proof of the forward direction of the following (for details see Marker, Section 5.2 or Tent-Ziegler, 5.1-5.5).

Theorem (Baldwin-Lachlan)

Let κ be uncountable. A countable theory T is κ-categorical iff T is ω-stable and has no Vaughtian pairs.