36. Indiscernibles

We will show that structures generated by indiscernible elements realize only few types.

Let $\mathcal{A}$ be an $\mathcal{L}$-structure, let I be a linearly ordered set.
A sequence $(a_i)_{i\in I}$ of elements in A is a **sequence of order indiscernibles** if for all $n \in \mathbb{N}$, all $\mathcal{L}$-formulas $\phi(x_1, \ldots, x_n)$ and all $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ from I

$$\mathcal{A} \models \phi(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow \phi(a_{j_1}, \ldots, a_{j_n}).$$

- If $a_i = a_j$ for $i \neq j$ are equal, then $(a_i)_{i\in I}$ is constant.
- $(a_i)_{i\in I}$ is **totally indiscernible** if the order on I does not matter.
- Fact: T is stable iff order = totally indiscernible.

Example

1. In $(\mathbb{Q}, <)$ any strictly increasing (or decreasing) sequence of elements is a sequence of order (not totally) indiscernibles
2. In a vector space, any sequence of linearly independent elements is (totally) indiscernible.

Models with infinite sequences of indiscernibles exist

Let I an infinite linearly ordered set and $(a_i)_{i \in I}$ a sequence of elements in $\mathcal{A}$.
The **Ehrenfeucht-Mostowski type** $\mathrm{EM}((a_i)_{i \in I})$ is the set of formulas $\phi(x_1, \ldots, x_n)$ such that $\mathcal{M} \models \phi(a_{i_1}, \ldots, a_{i_n})$ holds for all $n \in \mathbb{N}$ and all $i_1 < \cdots < i_n$ in I.

Lemma (The Standard Lemma (Tent-Ziegler 5.1.3))

Let I, J be infinite linearly ordered sets, let $(a_i)_{i \in I}$ be a sequence of elements of $\mathcal{A}$. Then there exists $\mathcal{B} \equiv \mathcal{A}$ and an order indiscernible sequence $(b_j)_{j \in J}$ of elements of $\mathcal{B}$ realizing $\mathrm{EM}((a_i)_{i \in I})$.

Proof.

Let $(c_j)_{j\in J}$ be a sequence of new constants,

$$U := \{\phi(\bar{c}) \mid \phi(\bar{x}) \in \mathrm{EM}((a_i)_{i\in I}), \bar{c} \text{ a subsequence of } (c_j)\},$$

$$V := \{\phi(\bar{c}) \leftrightarrow \phi(\bar{d}) \mid \bar{c}, \bar{d} \text{ subsequences of } (c_j)\}.$$

To see that $\mathrm{Th}(\mathcal{A}) \cup U \cup V$ has a model $\mathcal{B}_C$, we show for any $n \in \mathbb{N}$, finite $C_0 \subseteq (c_j)_{j\in J}$ and finite set Δ of formulas $\phi(x_1, \ldots, x_n)$ that

$$T_{C_0,\Delta} := \mathrm{Th}(\mathcal{A}) \cup \{\phi(\bar{c}) \in U | \bar{c} \in C_0\} \cup \{\phi(\bar{c}) \leftrightarrow \phi(\bar{d}) | \phi \in \Delta, \bar{c}, \bar{d} \in C_0\}$$

is satisfiable.

Define an equivalence relation $\sim$ on n-subtuples of $(a_i)_{i\in I}$ by

$$\bar{a} \sim \bar{b} \text{ iff } \mathcal{A} \models \phi(\bar{a}) \leftrightarrow \phi(\bar{b}) \text{ for all } \phi \in \Delta.$$

Now $\sim$ has $\leq 2^{\Delta}$ many classes. By Ramsey's Theorem, there exists infinite $K \subseteq I$ such that all n-subtuples of $(a_i)_{i\in K}$ are in the same class. For $C_0 = \{c_{j_1}, \ldots, c_{j_\ell}\}$ with $j_1 < \cdots < j_\ell$, choose $k_1 < \cdots < k_\ell$ in K. Then $(\mathcal{A}, a_{k_1}, \ldots, a_{k_\ell}) \models T_{C_0,\Delta}$. □

Corollary (Tent-Ziegler, 5.1.4)

Let T be a theory with infinite models and I a linearly ordered set. Then there exists $\mathcal{A} \models T$ with an order indescernible sequence $(a_i)_{i \in I}$ of distinct elements in A.

Proof.

By Löwenheim-Skolem there exists a model with a sequence $(a_i)_{i \in I}$ of distinct elements. Then apply the Standard Lemma. □

We need some preparation to show that every countable theory has a model that realizes only countably many types over any countable set. We will use this to show that uncountably categorical theories are ω-stable.

Lemma (Tent-Ziegler, 5.1.6)

Let $\mathcal{A}$ be an $\mathcal{L}$-structure for $\mathcal{L}$ countable that is generated by a well-ordered sequence $(a_i)_{i \in I}$ of indiscernibles. Then $\mathcal{A}$ realizes only countably many types over any countable subset of A.

Proof sketch.

To simplify, assume $\mathcal{L}$ has no function symbols.
Let $B \subseteq A$ be countable, let $a \in A$.
Since the $(a)_i$ are indiscernible and well-ordered, $\mathrm{tp}(a/B)$ only depends on the position relative to the elements in B.
Hence there are only countably many types.

The general case is similar after writing a, B in terms of $(a_i)_{i \in I}$. □

Skolemization of theories

Add function symbols to a language $\mathcal{L} = \mathcal{L}_0$ to witness existential statements in a theory $T = T_0$:

- $\mathcal{L}_{i+1}$ is obtained from $\mathcal{L}_i$ by adding a new n-ary **Skolem function** f_ϕ for each quantifier-free $\mathcal{L}_i$-formula $\phi(x_1, \ldots, x_n, y)$.
- $T_{i+1} := T_i \cup \{\forall \bar{x} \ (\exists y \ \phi(\bar{x}, y) \rightarrow \phi(\bar{x}, f_\phi(\bar{x})) \mid \phi \text{ qf } \mathcal{L}_i\text{-formula}\}$.

Then $T^* := \bigcup_{i \in \mathbb{N}} T_i$ is a **skolemization of** T. Note that

- $\mathcal{L}^* := \bigcup_{i \in \mathbb{N}} \mathcal{L}_i$ has cardinality $|\mathcal{L}| + \aleph_0$.
- Every model $\mathcal{M}$ of T can be extended to a model $\mathcal{M}^*$ of T^*.
- T^* has quantifier elimination.
- For every $\mathcal{B}^* \models T^*$ and $\mathcal{A}^* \leq \mathcal{B}^*$ we have $\mathcal{A}^* \models T^*$ (i.e. T^* has a **universal axiomiatization**).

(see Marker 2.3.6, 2.3.9)

Corollary (Tent-Ziegler, 5.1.9)

Let T be a countable $\mathcal{L}$-theory with an infinite model and let κ be an infinite cardinal.
Then T has a model of cardinality κ that realizes only countably many types over any countable subset.

Proof.

Let T^* be a skolemization of T as above.
By the Standard Lemma, T^* has a model $\mathcal{B}^*$ with order indiscernibles $(a_i)_{i \in I}$ for a well-ordering I of κ.
Let $\mathcal{A}^* \leq \mathcal{B}^*$ be generated by $(a_i)_{i \in I}$. Then $|A^*| = \kappa$ and $\mathcal{A}^* \models T^*$.
Since T^* has quantifier elimination, $\mathcal{A}^* \prec \mathcal{B}^*$ and $(a_i)_{i \in I}$ are still indiscernibles in $\mathcal{A}^*$.
By the lemma above, $\mathcal{A}^*$ realizes only countably many types over any countable set, and the same holds for the $\mathcal{L}$-reduct $\mathcal{A}$. □