

## 35. Ramsey theory

# Some combinatorics

## Example

Every complete graph with  $\geq 6$  vertices whose edges are colored red or blue contains a monochromatic triangle.

Let  $[A]^n$  be the set of  $n$ -element subsets of  $A$ .

$\chi: [A]^n \rightarrow \{0, \dots, k-1\}$  is a **coloring (partition)** of the  $n$ -subsets of  $A$  in  $k$  colors.

## Theorem (Ramsey)

For every infinite  $A$  and every coloring of  $[A]^n$  in  $k$  colors, there exists an infinite  $B \subseteq A$  such that  $[B]^n$  is monochromatic.

## Proof (by induction on $n$ ).

For  $n = 1$ , this is the (infinite) pigeonhole principle.

Assume the theorem is true for  $n \geq 1$ .

1. Pick  $a_0 \in A$ . A  $k$ -coloring  $\chi$  of  $[A]^{n+1}$  yields a  $k$ -coloring  $\chi'$  of  $[A \setminus \{a_0\}]^n$  by  $\chi'(C) := \chi(C \cup \{a_0\})$ .
2. By induction assumption, we have an infinite  $B_1 \subseteq A \setminus \{a_0\}$  with  $[B_1]^n$ , i.e.  $\{C \cup \{a_0\} \mid C \in [B_1]^n\}$  monochromatic.
3. Pick  $a_1 \in B_1$  and repeat to obtain infinite  $B_2 \subseteq B_1 \setminus \{a_1\}$  with  $\{C \cup \{a_1\} \mid C \in [B_2]^n\}$  monochromatic.
4. Inductively we construct

$$A = B_0 \supset B_1 \supset B_2 \supset \dots$$

and  $a_i \in B_i \setminus B_{i+1}$  such that the color of each  $(n+1)$ -element subset  $\{a_{i(0)}, \dots, a_{i(n)}\}$  with  $i(0) < \dots < i(n)$  only depends on  $i(0)$ .

5. By the pigeonhole principle, there are infinitely many values  $i(0)$  determining the same color. These  $a_{i(0)}$  yield the required monochromatic set. □

As a consequence we get

### Finite Ramsey Theorem

For all  $k, n, m \in \mathbb{N}$  there exists  $\ell \in \mathbb{N}$  such that every  $k$ -coloring of  $n$ -element subsets of  $A$  with  $|A| = \ell$  has a monochromatic subset  $B$  with  $|B| = m$ .

In arrow notation,

$$\ell \rightarrow (m)_k^n.$$

### Ramsey theory idea

Every partition of a large enough object has a regular part.