34. Binary trees of formulas

Binary trees

For constructing $2^{\aleph_0}$ types, we consider trees.

Example

Let $\mathcal{L}$ be the language with a unary predicate P_s for every finite 0/1-sequence $s \in \{0,1\}^{<\omega} := \bigcup_{n<\omega}\{0,1\}^n$.
The axioms of **Tree** say that the P_s form a binary decomposition of the universe:

- $\forall x\ P_\emptyset(x)$
- $\exists x\ P_s(x)$
- $\forall x\ (P_{s0}(x) \vee P_{s1}(x)) \leftrightarrow P_s(x)$
- $\forall x\ \neg(P_{s0}(x) \wedge P_{s1}(x))$.

Model: Cantor space 2^ω with P_s the set of sequences starting in s.
Tree has quantifier elimination (any $\phi(x)$ is equivalent to $\bigvee P_{s_i}(x)$ for some s_i) and is complete.
Tree has no isolated types and no prime model.

A theory T **has a binary tree of formulas** $\varphi_s(\bar{x})$ for $s \in \{0,1\}^{<\omega}$ if for every $s \in \{0,1\}^{<\omega}$:

- $T \models \forall \bar{x} \ (\phi_{s0}(\bar{x}) \vee \phi_{s1}(\bar{x})) \rightarrow \phi_s(\bar{x})$,
- $T \models \forall \bar{x} \ \neg(\phi_{s0}(\bar{x}) \wedge \phi_{s1}(\bar{x}))$.

Theorem

Let T be a complete theory.

1. If $|S_n(T)| < 2^{\aleph_0}$ for all $n \in \mathbb{N}$, then T has no tree of consistent formulas.
2. If T has no tree of consistent formulas, then the isolated types are dense. If additionally T is countable, then $|S_n(T)| \leq \aleph_0$.

Proof 1 (by contraposition).

Let $\phi_s(\bar{x})$ for $s \in \{0,1\}^{<\omega}$ be a binary tree of consistent formulas. Then for every $\sigma \in \{0,1\}^{\omega}$ the set

$$\{\phi_s(\bar{x}) \mid s \text{ is a prefix of } \sigma\}$$

is satisfiable and hence in some complete type $p_\sigma(x) \in S_n(T)$. Since the p_σ are all distinct, $|S_n(T)| = 2^{\aleph_0}$.

Proof 2 (second part by contraposition).

Assume T is countable and $|S_n(T)| > \aleph_0$.
By compactness, $S_n(T)$ is covered by finitely many basic clopens $[\phi]$.
So some $|[\phi]| > \aleph_0$.

Claim. If $|[\phi]| > \aleph_0$, then there exists ψ such that $|[\phi \wedge \psi]| > \aleph_0$ and $|[\phi \wedge \neg\psi]| > \aleph_0$.
[Otherwise, $p := \{\psi \mid |[\phi \wedge \psi]| > \aleph_0\}$ is satisfiable because

1. For $\psi_1, \ldots, \psi_k \in p$, either $\bigwedge \psi_i \in p$ or $\bigvee \neg\psi_i \in p$.
2. In the former case $T \cup \{\psi_1, \ldots, \psi_k\}$ is satisfiable.
3. In the latter $[\phi \wedge \bigvee \neg\psi_i] = \bigcup [\phi \wedge \neg\psi_i]$ implies $|[\phi \wedge \neg\psi_i]| > \aleph_0$ for some $i \leq k$, a contradiction.

So p is a complete type.
Since any type $q \neq p$ contains some $\psi \notin p$,

$$[\phi] = \bigcup_{\psi \notin p} [\phi \wedge \psi] \cup \{p\},$$

which is countable, a contradiction.]

We inductively construct a binary tree ϕ_s for $s \in 2^{<\omega}$ of consistent formulas such that all $|[\phi_s]| > \aleph_0$:

- Given ϕ_s, pick ψ as in the claim above and set

$$\phi_{s0} := \phi_s \wedge \psi, \quad \phi_{s1} := \phi_s \wedge \neg\psi.$$

Hence the second part of 2 is proved.

For the proof of the density of isolated types see Marker, Thm 4.2.11 (i), or Tent-Ziegler, Thm 4.5.9(2). □

ω-stable $\Rightarrow$ κ-stable for all κ

Theorem

If a complete theory T in a countable language is ω-stable, then T is κ-stable for all infinite κ.

Proof (by contraposition).

Let $\mathcal{M} \models T, A \subseteq M, |A| = \kappa$, and $|S_n^{\mathcal{M}}(A)| > \kappa$.
Note that there are only κ formulas over $\mathcal{L}_A$.
So, as in the previous proof, we have some $|[\phi_\emptyset]| > \kappa$ and can construct a binary tree ϕ_s for $s \in 2^{<\omega}$ of consistent formulas such that all $|[\phi_s]| > \kappa$.
The subset $A_0 \subseteq A$ of parameters occurring in all ϕ_s is countable.
By the previous Theorem (part 1), $|S_n^{\mathcal{M}}(A_0)| = 2^{\aleph_0}$ and T is not ω-stable. □