33. Stability

Given a theory T we may want to

- classify its models,
- understand properties of elements,
- understand definable sets ...

We have seen how n-types with parameters A apply to these.
In principle $|S_n(A)|$ can be $2^{|A|}$. We bound the cardinality further.

Convention. Let T be a complete theory with infinite models, let κ be an infinite cardinal.

T is κ-**stable** if for every $\mathcal{M} \models T$ and every $A \subseteq M$ with $|A| \leq \kappa$, we have $|S_n^{\mathcal{M}}(A)| \leq \kappa$ for all $n \in \mathbb{N}$.
T is **stable** if T is κ-stable for some κ; else **unstable**.
A structure is **stable** if its theory is.

Example

- DLO is not ω-stable.
- ACF_p for p prime or 0 is κ-stable for every κ.

Stable = stable for 1-types

Lemma

T is κ-stable iff $|S_1^{\mathcal{M}}(A)| \leq \kappa$ for all $\mathcal{M} \models T, A \subseteq M, |A| \leq \kappa$.

$\Leftarrow$ by induction on n

Let $\mathcal{M} \models T, A \subseteq M, |A| \leq \kappa$. We have $\mathcal{N} \succ \mathcal{M}$ such that $\mathcal{N}$ realizes all 1-types over A (Realizing Types Lemma, Slides 18).
For $p \in S_n^{\mathcal{M}}(A)$, define the projected 1-type

$$p' := \{\exists x_1, \ldots, x_{n-1}\ \phi(x_1, \ldots, x_n) \mid \phi \in p\} \in S_1^{\mathcal{M}}(A).$$

Assume $p' = \mathrm{tp}^{\mathcal{N}}(b/A)$ for some $b \in N$. Then

$$\{q \in S_n^{\mathcal{M}}(A) | q' = p'\} \to S_{n-1}^{\mathcal{N}}(A \cup \{b\}),\ q \mapsto \{\phi(x_1, \ldots, x_{n-1}, b) | \phi \in q\},$$

is a bijection (Tent-Ziegler, Exercise 4.2.5).
Hence by induction assumption

$$|S_n^{\mathcal{M}}(A)| \leq |S_1^{\mathcal{M}}(\mathcal{A})| \cdot |S_{n-1}^{\mathcal{N}}(\mathcal{A} \cup \{b\})| \leq \kappa \cdot \kappa = \kappa. \qquad \square$$

Outlook: Stability spectrum for countable theories

Theorem (Shelah)

Any countable complete theory T satisfies exactly one of the following:

- T is κ-stable for all infinite κ (totally transcendental[1]);
- T is κ-stable iff $\kappa \geq 2^{\aleph_0}$ (superstable but not totally transcendental);
- T is κ-stable iff $\kappa = \kappa^{\aleph_0}$ (stable but not superstable);
- T is unstable.

[1] Terminology due to Morley: every type has ordinal-valued Morley rank.