31. ω-categorical structures

Oligomorphic permutation groups

A permutation group G on a countably infinite set X is **oligomorphic** if G has only finitely many orbits of n-tuples for each $n \geq 1$.

Example

1. $\mathrm{Sym}\,\mathbb{N}$ is the group of all bijections on $\mathbb{N}$.
   Orbits on $\mathbb{N}^2$
2. $\mathrm{Aut}\,(\mathbb{Q}, <)$ is the group of order preserving functions on $\mathbb{Q}$; oligomorphic since $(\mathbb{Q}, <)$ is homogeneous.

Characterization of ω-categorical structures

Theorem (Engeler, Ryll-Nardzewski, Svenonius)

For a countably infinite structure $\mathcal{A}$ TFAE:

1. $\mathcal{A}$ is ω-categorical.
2. Every type of $\mathrm{Th}(\mathcal{A})$ is isolated.
3. Every model of $\mathrm{Th}(\mathcal{A})$ is atomic.
4. $S_n(\mathrm{Th}(\mathcal{A}))$ is finite for all $n \geq 1$.
5. For each $n \geq 1$, there are only finitely many inequivalent formulas with free variables $x_1, \ldots, x_n$ modulo $\mathrm{Th}(\mathcal{A})$.
6. Every model of $\mathrm{Th}(\mathcal{A})$ is ω-saturated.
7. The automorphism group $\mathrm{Aut}\,(\mathcal{A})$ of $\mathcal{A}$ is oligomorphic.

Atomic cycle 1⇒2⇒3⇒1

Let $T := \mathrm{Th}(\mathcal{A})$.

1⇒2 by contraposition. If T has a non-isolated type p, then also a countable model omitting p (Omitting Types Thm).
Since T also has a countable model realizing p (Löwenheim-Skolem), it is not ω-categorical.

2⇒3. By definition of atomic.

3⇒1. follows since all countable atomic models are isomorphic.

Saturated cycle 2⇒4⇒5⇒6⇒1

2⇒4. Assume each $p \in S_n(T)$ is isolated by some ϕ_i for $i \in I$.
Then $T \cup \{\neg\phi_i \mid i \in I\}$ is not satisfiable.
By the Compactness Thm, there exists a finite $F \subseteq I$ such that
$T \cup \{\neg\phi_i \mid i \in I\}$ is not satisfiable, i.e.,
in every $\mathcal{B} \models T$ every $\bar{b} \in B^n$ satisfies ϕ_i for some $i \in F$.
Hence $|S_n(\mathcal{A})| \leq |F|$.

4⇒5. Modulo T, any formula $\phi(\bar{x})$ is uniquely determined by
$[\phi] := \{p \in S_n(T) \mid \phi \in T\}$ (HW 7).
If $S_n(T)$ is finite, there are only finitely many clopens $[\phi]$, i.e.,
formulas mod T.

5⇒6. Let $\mathcal{B} \models T$, $\bar{b} \in B^m$ and $p \in S_1^{\mathcal{B}}(\bar{b})$.
Let $\phi_1(x, \bar{y}), \ldots, \phi_k(x, \bar{y})$ be representatives for all formulas mod T.
Then $\bigwedge\{\phi_i(x, \bar{b}) \ : \ \phi_i(x, \bar{b}) \in p, i \leq k\}$ isolates p, which must
then be realized.

6⇒1. follows from the uniqueness of saturated models.

Automorphism cycle 2∧4⇒7⇒5.

2∧4⇒7. Recall (slides 23) that in a countable atomic $\mathcal{A}$

$$\mathrm{tp}^{\mathcal{A}}(\bar{a}_1) = \mathrm{tp}^{\mathcal{A}}(\bar{a}_2) \text{ iff } \exists h \in \mathrm{Aut}\,\mathcal{A} : h(\bar{a}_1) = \bar{a}_2.$$

Hence $|S_n(T)| = |\text{orbits of } \mathrm{Aut}\,\mathcal{A} \text{ on } A^n|$ is finite.

7⇒5. Since formulas are preserved by automorphisms, $\mathcal{A}$ realizes only finitely many n-types, say $p_1, \ldots, p_k$. Since types are filters, for each distinct $i, j \leq k$ there exist $\phi_i \in p_i \setminus p_j$. So

$$\mathcal{A} \models \forall \bar{x} \bigvee_{i=1}^{k} \phi_i(\bar{x}).$$

For any ψ,

$$\mathcal{A} \models \forall \bar{x} \forall \bar{y}\ (\phi_i(\bar{x}) \wedge \phi_i(\bar{y})) \rightarrow (\psi(\bar{x}) \leftrightarrow \psi(\bar{y})).$$

Hence for any $\mathcal{B} \models T$ and $\bar{b} \in B^n$ there is some $i \leq k$ such that $\mathcal{B} \models \phi_i(\bar{b})$. If $\mathcal{B} \models \phi_i(\bar{b}), \phi_i(\bar{c})$, then $\mathrm{tp}^{\mathcal{B}}(\bar{b}) = \mathrm{tp}^{\mathcal{B}}(\bar{c})$. So $\{p_1, \ldots, p_k\} = S_n(T)$.

□

ω-categorical $\Rightarrow$ uniformly locally finite

Corollary

Every ω-categorical $\mathcal{A}$ is **uniformly locally finite**, i.e. there exists $f\colon \mathbb{N} \to \mathbb{N}$ such that every n-generated substructure of $\mathcal{A}$ has size $\leq f(n)$

Proof.

For $\bar{a} \in A^n$ let $b, c \in \langle \bar{a} \rangle \leq \mathcal{A}$ be distinct.
Then $\mathrm{tp}^{\mathcal{A}}(\bar{a}, b) \neq \mathrm{tp}^{\mathcal{A}}(\bar{a}, c)$.
So $|\langle \bar{a} \rangle| \leq |S_{n+1}(\mathrm{Th}(\mathcal{A}))|$, which is finite by Ryll-Nardzewski. □

Example

If a countable group G is ω-categorical, then it is locally finite and has finite exponent (converse holds for abelian G).

Homogeneous+uniformly loc fin $\Leftrightarrow$ ω-categorical+QE

Theorem (Hodges 7.4.2)

For $\mathcal{A}$ countable over a finite language TFAE:

1. $\mathcal{A}$ is (ultra)homogeneous and uniformly locally finite;
2. $\mathrm{Th}(\mathcal{A})$ is ω-categorical and has quantifier elimination.

Proof sketch.

2⇒1. For homogeneity, let $h\colon \langle \bar{a}_1 \rangle \to \langle \bar{a}_2 \rangle$ be an isomorphism with $h(\bar{a}_1) = \bar{a}_2$. Then $\bar{a}_1, \bar{a}_2$ satisfy the same quantifier-free formulas.
Hence $\mathrm{tp}(\bar{a}_1) = \mathrm{tp}(a_2)$ by quantifier elimination.
By ω-categoricity, this is equivalent to $\bar{a}_1, \bar{a}_2$ being in the same $\mathrm{Aut}\,(\mathcal{A})$-orbit, i.e. h extends to an automorphism of $\mathcal{A}$.

Uniformly local finiteness follows from the previous Corollary.

1⇒2. By the cardinality bound on n-generated substructures and finite language, $\mathcal{A}$ has $g(n)$ many n-generated substructures up to isomorphism for each $n \in \mathbb{N}$.
So $\mathrm{Aut}(\mathcal{A})$ has $g(n)$ orbits on A^n by homogeneity.
Thus $\mathcal{A}$ is ω-categorical by Ryll-Nardzewski.

Homogeneity yields quantifier elimination (omitted). □

Examples

Which are (ultra)homogeneous, ω-categorical?

1. $(\mathbb{Q}, <)$
2. Rado graph
3. homogeneous universal partial order
4. countably dimensional $\mathcal{F}$-vector space
5. algebraic closure of prime field $\mathcal{F}_p$ for p prime or $p = 0$
6. countable atomless Boolean algebra
7. Hall's universal group
8. countable graph with exactly one edge