30. Countable saturated models

Countable saturated models

Lemma

$\mathcal{A} \models T$ is $\aleph_0$-saturated iff $\mathcal{A}$ is $\aleph_0$-homogeneous and realizes all types in $S_n(T)$ for all $n \in \mathbb{N}$.

Proof.

$\Rightarrow$: clear.
$\Leftarrow$: For $\bar{b} \in A^n$ and $p \in S_n^{\mathcal{A}}(\bar{b})$, let

$$q := \{\phi(\bar{x}, \bar{y}) \mid \phi(\bar{x}, \bar{b}) \in p\} \in S_{m+n}(T).$$

By assumption we have $(\bar{a}, \bar{b}') \in A^{m+n}$ realizing q.
Now $\mathrm{tp}^{\mathcal{A}}(\bar{b}') = \mathrm{tp}^{\mathcal{A}}(\bar{b})$ means there is a partial elementary map f from $\bar{b}'$ to $\bar{b}$.
Since $\mathcal{A}$ is $\aleph_0$-homogeneous, f extends mapping $\bar{a}$ to some $\bar{a}' \in A^m$ such that $\mathrm{tp}^{\mathcal{A}}(\bar{a}, \bar{b}') = \mathrm{tp}^{\mathcal{A}}(\bar{a}', \bar{b})$.
Hence $\bar{a}'$ realizes p. $\square$

Lemma

Every $\mathcal{A}$ has an $\aleph_0$-homogenous elementary extension $\mathcal{B}$ such that $|A| = |B|$.

Proof.

Construct $\mathcal{B}$ as limit of elementary chains, see Marker, Prop 4.3.6. $\square$

Existence of saturated models

Theorem

TFAE for T:

1. T has a saturated countable model.
2. T has a universal countable model.
3. $|S_n(T)| \leq \aleph_0$ for all $n \in \mathbb{N}$.

Proof.

2⇒3. Assume T has a universal countable model $\mathcal{A}$.
Every $p(\bar{x}) \in S_n(T)$ is realized in some countable model $\mathcal{B}$, say by $\bar{b}$.
For $f : \mathcal{B} \to \mathcal{A}$ elementary, $f(\bar{b})$ realizes p in $\mathcal{A}$.
Thus $\mathcal{A}$ realizes every type in $S_n(T)$.
Since A^n is countable, so is $S_n(T)$.

3⇒1. Enumerate $\bigcup_{n \in \mathbb{N}} S_n(T)$ as $p_0, p_1, \ldots$.
Let $\mathcal{A}_0 \models T$ be countable.
Iterate the Realizing Types Lemma (Slides 18) to construct

$$\mathcal{A}_0 \prec \mathcal{A}_1 \prec \ldots$$

such that $\mathcal{A}_{i+1}$ is countable and realizes p_i.
Then $\mathcal{A} := \lim_{i \in \mathbb{N}} A_i$ is countable and realizes all types in $S_n(T)$.
By Lemma above, there exists $\mathcal{A} \prec \mathcal{B}$ such that $\mathcal{B}$ is countable and $\aleph_0$-homogeneous.
By penultimate Lemma, $\mathcal{B}$ is $\aleph_0$-saturated. □

Corollary (Marker 4.3.8)

1. If T has a countable saturated model, then it has a prime model.
2. If T has $< 2^{\aleph_0}$ many countable models, then T has a countable saturated model.

Example (RCF)

RCF has $2^{\aleph_0}$ many 1-types (cuts over $\mathbb{Q}$), hence no saturated model.