# 29. Homogeneous and universal models

# $\kappa$-homogeneity

A model $\mathcal{A} \models T$ is $\kappa$-**homogeneous** if for any $B \subseteq A$ with $|B| < \kappa$, any partial elementary map $f \colon B \to A$ and any $a \in A$ there exists an extension $f^*$ of $f$ such that $f^* \colon B \cup \{a\} \to A$ is partial elementary.
$\mathcal{A}$ is **homogenous** if it is $|A|$-homogenous.

- Warning: This is not Fraïssé's (ultra)homogeneity on finitely generated substructures.
- Still, $\kappa$-categorical implies $\kappa$-homogeneous (later).
- In homogeneous models, partial elementary maps are just restrictions of automorphisms [see Marker, Prop 4.2.13, for a back-and-forth argument].
- Countable atomic models are homogeneous [Marker, Lemma 4.2.14].

# saturated $\Rightarrow$ homogeneous

## Lemma

If $\mathcal{A}$ is $\kappa$-saturated, then $\mathcal{A}$ is $\kappa$-homogeneous.

## Proof.

For $p := \mathrm{tp}(a/B)$, let

$$f(p) := \{\varphi(x, f(\bar{b})) \mid \mathcal{A} \models \varphi(a, \bar{b}), \bar{b} \text{ a tuple over } B\}$$

be the corresponding type over $f(B)$.
Note that $f(p)$ is satisfiable (HW).
Since $\mathcal{A}$ is saturated, some $b \in A$ realizes $f(p)$.
Then $f^* := f \cup \{(a, b)\}$ is the required extension. $\square$

# Uniqueness of saturated models

## Theorem

Let $\mathcal{A}, \mathcal{B}$ be saturated such that $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$ and $|A| = |B|$.
Then $\mathcal{A} \cong \mathcal{B}$.

## Proof by back-and-forth argument.

Enumerate $A, B$ as $(a_\alpha)_{\alpha<\kappa}$, $(b_\alpha)_{\alpha<\kappa}$.
By transfinite induction, construct an increasing sequence of (partial) elementary maps $f_\alpha \colon A_\alpha \to B_\alpha$.
Assume $f_\beta$ is constructed for all $\beta < \alpha$. The union

$$f^*_\alpha := \bigcup_{\beta<\alpha} f_\beta : A^*_\alpha \to B^*_\alpha \text{ is elementary.}$$

Write $\alpha = \lambda + n$ for $\lambda$ a limit ordinal or 0 and $n \in \mathbb{N}$.

$n = 2i$ **(forth):** As in the previous lemma, let $p := \mathrm{tp}(a_{\lambda+i}/A^*_\alpha)$. Since $\mathcal{B}$ is saturated, some $b \in B$ realizes $f^*_\alpha(p)$. Define

$$f_\alpha := f^*_\alpha \cup \{(a_{\lambda+i}, b)\}.$$

$n = 2i + 1$ **(back):** Similarly, using that $\mathcal{A}$ is saturated, extend

$$f_\alpha := f^*_\alpha \cup \{(a, b_{\lambda+i})\}.$$

Note that $|A^*_\alpha|, |B^*_\alpha| \leq |\alpha| < \kappa$ by construction.

Finally $\bigcup_{\alpha<k} f_\alpha$ is the required isomorphism from $\mathcal{A}$ to $\mathcal{B}$. $\square$

# Saturated models embed all small models

A model $\mathcal{A} \models T$ is $\kappa$-**universal** if for every $\mathcal{B} \models T$ with $|B| < \kappa$ there exists an elementary embedding $\mathcal{B} \to \mathcal{A}$.
$\mathcal{A}$ is **universal** if it is $|A|^+$-universal.

(Recall from set theory: If $\kappa$ is a cardinal, then there is a least cardinal greater than $\kappa$, denoted $\kappa^+$.)

## Lemma

Let $\kappa$ be infinite. If $\mathcal{A}$ is $\kappa$-saturated, then $\mathcal{A}$ is $\kappa^+$-universal.
In particular, if $\mathcal{A}$ is a saturated countable model, then any countable model elementary embeds into $\mathcal{A}$.

## Proof.

Forth. □

# saturated = homogenous+universal

## Theorem

For $\kappa$ infinite TFAE:

1. $\mathcal{A}$ is $\kappa$-saturated.
2. $\mathcal{A}$ is $\kappa$-homogeneous and $\kappa^+$-universal.

If $\kappa \geq \aleph_1$, then 1, 2. are also equivalent to

3. $\mathcal{A}$ is $\kappa$-homogeneous and $\kappa$-universal.

## Proof.

It remains to show 2⇒1. (3⇒1. if $\kappa \geq \aleph_1$).
Let $B \subseteq A$ with $|B| \leq \kappa$, let $p \in S_1^{\mathcal{A}}(B)$.
We have $\mathcal{C} \models \mathrm{Th}(\mathcal{A}_B)$ with $B \subseteq C$ and $c \in C$ realizing $p$ where

- $|C| = \aleph_0$ if $\kappa = \aleph_0$;
- $|C| < \kappa$ infinite if $\kappa \geq \aleph_1$.

Since $\mathcal{A}$ is universal, we have elementary $f : \mathcal{C} \to \mathcal{A}$.
Since $\mathcal{A}$ is $\kappa$-homogeneous, there is $a \in A$ such that

$$\mathrm{tp}^{\mathcal{A}}(a/B) = \mathrm{tp}^{\mathcal{A}}(f(c)/f(B)) = \mathrm{tp}^{\mathcal{C}}(c/B) = p$$

Thus $\mathcal{A}$ is $\kappa$-saturated. □

## Corollary

$\mathcal{A}$ is saturated iff $\mathcal{A}$ is homogenous and universal.