

28. Saturated models

Convention. Let T be a complete theory with infinite models over a countable language \mathcal{L} .

Recall: Prime models realize as few types as possible.

We now consider models that realize as many types as possible:

For an infinite cardinal κ , a structure \mathcal{A} is **κ -saturated** if \mathcal{A}_B realizes all 1-types over B for all $B \subseteq A$ with $|B| < \kappa$.

An infinite structure \mathcal{A} is **saturated** if it is $|A|$ -saturated.

Note

- ▶ The condition $< |A|$ is essential: the (partial) type $\{x \neq a \mid a \in A\}$ is not realized in \mathcal{A} .
- ▶ Types over parameters B are needed to talk e.g. about transcendental elements.
- ▶ If \mathcal{A}_B realizes all $p \in S_1^{\mathcal{A}}(B)$, then also all $p \in S_n^{\mathcal{A}}(B)$ for all $n \in \mathbb{N}$ (HW).

Examples

$(\mathbb{Q}, <)$ is saturated.

By homogeneity, there are exactly $2k + 1$ types over $a_1 < \dots < a_k$ in \mathbb{Q} . Each is realized in \mathbb{Q} .

$(\mathbb{R}, <)$ is not saturated.

$\{0 < x\} \cup \{x < \frac{1}{n} \mid n \in \mathbb{N}\}$ is not realized in \mathbb{R} .

$(\mathbb{R}, +, -, \cdot, 0, 1)$

The algebraic closure \mathcal{F} of the field $\mathbb{Q}(x_1, x_2, \dots)$ is saturated.

Let $B \subseteq F$, finite, generate a subfield \mathcal{K} of \mathcal{F} .

Since ACF has quantifier elimination, any \mathcal{L}_B -formula is equivalent to a Boolean combination of formulas $f(x) = 0$ for $f \in K[x]$.

- ▶ If a type $p(x)$ contains some formula $f(x) = 0$, then it is **algebraic** over \mathcal{K} and realized in \mathcal{F} .
- ▶ Else $p(x) = \{f(x) \neq 0 \mid f \in K[x], f \neq 0\}$ (by completeness) is the **unique transcendental type** over B and realized by any $a \in F \setminus \overline{K}$ (exists since \overline{K} has fin transcendence degree over \mathbb{Q}).

$(\mathbb{C}, +, -, \cdot, 0, 1)$