27. Consequences of quantifier elimination

Consequences of quantifier elimination

A theory T is **model-complete** if for all models $\mathcal{A}, \mathcal{B}$ of T

$$\mathcal{A} \leq \mathcal{B} \quad \Rightarrow \quad \mathcal{A} \prec \mathcal{B}$$

T is model complete iff all embeddings are elementary.

Theorem

If T has quantifier elimination, then T is model complete.

Proof.

Let $\mathcal{A}, \mathcal{B} \models T$ and $\mathcal{A} \leq \mathcal{B}$.
Let ϕ be an arbitrary formula, equivalent modulo T to some quantifier-free ψ, let $\bar{a} \in A$. Then

$$\mathcal{A} \models \phi(\bar{a}) \quad \text{iff} \quad \mathcal{A} \models \psi(\bar{a}) \quad \text{iff} \quad \mathcal{B} \models \psi(\bar{a}) \quad \text{iff} \quad \mathcal{B} \models \phi(\bar{a}).$$

Recall quantifier-free formulas are preserved by substructures, extensions!
Hence $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. $\square$

Algebraically closed fields

We already proved that ACF_p are the completions of ACF by Tarski-Vaught Test (slides 12).

Corollary

ACF is model complete and ACF_p is complete for $p = 0$ or prime.

Proof.

Let $\mathcal{A}, \mathcal{B} \models \mathrm{ACF}_p$, let ϕ be a sentence in the language of rings.
We have a quantifier-free sentence ψ such that $\mathrm{ACF}_p \models \phi \leftrightarrow \psi$.
Since ψ is preserved by substructures and extensions,

$$\mathcal{A} \models \phi \quad \text{iff} \quad \mathcal{A} \models \psi \quad \text{iff} \quad \mathcal{F}_p \models \psi \quad \text{iff} \quad \mathcal{B} \models \psi \quad \text{iff} \quad \mathcal{B} \models \phi,$$

where $\mathcal{F}_p$ is the field of size p if $p > 0$; else $\mathbb{Q}$.
Thus $\mathcal{A} \equiv \mathcal{B}$ and ACF_p is complete. $\square$

Decidability

Recall: Every complete recursively enumerable first-order theory T is decidable.
[To decide $T \models \phi$, search for a proof of ϕ and of $\neg\phi$ in parallel.]

Corollary

ACF_p is decidable for $p = 0$ or prime.

In particular $\mathrm{Th}(\mathbb{C}, +, \cdot)$ is decidable.

Corollary

ACF is decidable.

Proof.

To decide ACF$\models \phi$, search for a proof of ACF$\vdash \phi$ or for a prime p and a proof of $\mathsf{ACF}_p \vdash \neg\phi$. □

Definable sets

What are the definable sets in an algebraically closed field $\mathcal{F}$?

Boolean combinations of $p(x_1, \ldots, x_n) = 0$ for $p \in F[x_1, \ldots, x_n]$.

Corollary (Strong Minimality)

Let $\mathcal{F}$ be an algebraically closed field and $X \subseteq F$ definable.
Then either X or its complement $F \setminus X$ is finite.
In particular, $\mathbb{R}$ is not definable in $\mathcal{C}$.

Proof.

Follows since sets $\{x \in F \mid p(x) = 0\}$ for $p \in F[x], p \neq 0$, are finite. $\square$

The theory of $(\mathbb{R}, +, \cdot)$

$(\mathbb{R}, +, -, \cdot, 0, 1)$ in the language of rings $\mathcal{L}_{\text{Ring}} = \{+, -, \cdot, 0, 1\}$ does not have quantifier elimination.

1. Else the field of reals would be strongly minimal as above; every definable subset would be either finite or cofinite.
2. But $\exists y \; x = y^2$ defines $\{x \in \mathbb{R} \mid 0 < x\}$ which is neither.
3. It turns out the order $<$ is the only obstruction to quantifier elimination.

Real closed fields (RCF)

We collect axioms (RCF) in the language of ordered rings $\mathcal{L}_{\mathrm{ORing}} = \{+, -, \cdot, 0, 1, <\}$ that hold for $\mathbb{R}$.

$\mathcal{F} = (F, +, \cdot, 0, 1, <)$ is a **real closed field** if

1. $\mathcal{F}$ is an **ordered field**, i.e., a field with a linear order $<$ satisfying
   - $\forall x, y, z \;\; x < y \to x + z < y + z$,
   - $\forall x, y \;\; (0 < x \wedge 0 < y) \to 0 < xy$;
2. (sign change) If $f(x) \in F[x]$ and $a < b$ in F are such that $f(a)f(b) < 0$, then there exists $a < c < b$ such that $f(c) = 0$.

[Sign change is expressible by first order sentences $\phi_1, \phi_2, \ldots$ for each degree n.]

Quantifier elimination of real closed fields

Theorem (Tarski)

RCF has quantifier elimination.

Proof (Marker, Section 3.3.)

Follows closely the proof for algebraically closed fields.

The key algebraic fact needed is that every ordered field has a unique real closure. □

Corollary

RCF is model complete, complete, decidable, the theory of $(\mathbb{R}, +, -, \cdot, 0, 1, <)$.

Proof.

By quantifier elimination, RCF is model complete.
Every real closed field $\mathcal{F}$ has characteristic 0, hence embeds $\mathbb{Q}$.
Then $\mathcal{F}$ embeds the field of real algebraic numbers $\mathbb{R}_{\text{alg}}$, so $\mathbb{R}_{\text{alg}} \prec \mathcal{F}$.
In particular $\mathcal{F} \equiv \mathbb{R}_{\text{alg}} \equiv \mathbb{R}$.
Thus RCF is complete, etc. □

Semialgebraic sets

What are the quantifier-free definable sets in a real closed field $\mathcal{F}$?
Boolean combinations of $p(x_1, \ldots, x_n) = 0$ and $q(x_1, \ldots, x_n) > 0$ for $p, q \in F[x_1, ..., x_n]$.
In real algebraic geometry these are known as **semialgebraic sets**.

$$\text{definable} = \text{quantifier-free definable} = \text{semialgebraic}$$

Corollary

Any definable subset of $\mathbb{R}$ is a finite union of points and intervals.
In particular, $\mathbb{Z}$ is not definable in $\mathbb{R}$.