24. Quantifier elimination

An $\mathcal{L}$-theory T has **quantifier elimination** if every $\mathcal{L}$-formula ϕ is equivalent modulo T to some quantfier-free formula ψ, i.e.,

$$T \models \forall \bar{x}\ \phi(\bar{x}) \leftrightarrow \psi(\bar{x}).$$

Application

If T has quantifier elimination, then every definable set is a Boolean combination of sets defined by atomic formulas.

Example

Many interesting theories have quantifier elimination:

- DLO, Rado graph, any Fraïssé limit in a fin relational language
- $(\mathbb{C}, +, -, \cdot, 0, 1)$, more generally ACF
- $(\mathbb{R}, +, -, \cdot, 0, 1, <)$, more generally RCF, the theory of real closed fields in the language of ordered rings
  (A field F is **real closed** if -1 is not a sum of squares and $F(i)$ with $i^2 = -1$ is algebraically closed.)

Non-examples

- $(\mathbb{R}, +, -, \cdot, 0, 1)$, RCF in the language of rings

- If T has quantifier elimination and $\mathcal{L}$ has no constants, then every sentence is equivalent to $\top$ or $\bot$. Hence T is either inconsistent or complete.
- Any theory T can be extended to a theory T^m (the **Morleyisation** of T) with quantifier elimination by expanding $\mathcal{L}$ by an n-ary relation symbol R_ϕ for any formula $\phi(x_1, \ldots, x_n)$ and T by

$$\forall \bar{x}\ R_\phi(x) \leftrightarrow \phi(\bar{x}).$$

T^m is complete, κ-categorical…iff T is.

Lemma

T has quantifier elimination iff every for every quantifier-free $\psi(y, \bar{x})$, the formula $\exists y\ \psi(y, \bar{x})$ is equivalent modulo T to a quantifier-free formula.

Proof.

$\Leftarrow$ Show for every $\phi(\bar{x})$ there exists quantifier-free $\theta(\bar{x})$ such that

$$T \models \forall \bar{x}\ \phi(\bar{x}) \leftrightarrow \theta(\bar{x})$$

by induction on ϕ.

Assume ψ_i is equivalent modulo T to some quantifier-free θ_i for $i = 0, 1$.

- If $\phi = \neg\psi_0$, then $T \models \forall \bar{x}\ \phi(\bar{x}) \leftrightarrow \neg\theta_0(\bar{x})$.
- If $\phi = \psi_0 \wedge \psi_1$, then $T \models \forall \bar{x}\ \phi(\bar{x}) \leftrightarrow \theta_0(\bar{x}) \wedge \theta_1(\bar{x})$.
- If $\phi(\bar{x}) = \exists y\ \psi_0(y, \bar{x})$, then $T \models \forall \bar{x}\ (\phi(\bar{x}) \leftrightarrow \exists y\ \theta_0(y, \bar{x}))$ and the latter is equivalent to quantifier-free formula by assumption. □

Quantifier-free formulas are preserved by isomorphisms

Lemma

ϕ is equivalent modulo T to some quantifier-free formula iff ϕ is preserved by isomorphisms between substructures of models of T, i.e., for all models $\mathcal{A}, \mathcal{B}$ of T, $\bar{a}$ over A and every embedding $h\colon \langle \bar{a} \rangle \to \mathcal{B}$,

$$\mathcal{A} \models \phi(\bar{a}) \quad \text{iff} \quad \mathcal{B} \models \phi(h(\bar{a})).$$

Proof.

$\Rightarrow$ (HW)

$\Leftarrow$ Let ϕ be preserved by isos between substructures of models.

We show that ϕ is equivalent to some formula in

$\Psi := \{\psi \text{ quantifier-free } \mid T \models \forall \bar{x}\ \phi(\bar{x}) \to \psi(\bar{x})\}$.

Claim 1: $T \cup \Psi \models \phi$.
Let $\mathcal{A} \models T$ and $\bar{a}$ over A such that $\mathcal{A} \models \Psi(\bar{a})$.
Let $\Sigma := \{\theta \text{ quantifier-free } \mid \mathcal{A} \models \theta(\bar{a})\}$.
Then $T \cup \Sigma \cup \{\phi\}$ is satisfiable.
[Else by the Compactness Thm there exists finite $\Delta \subseteq \Sigma$ such that $T \cup \Delta \cup \{\phi\}$ is not satisfiable, i.e. $T \cup \{\phi\}$ entails $\psi := \bigvee_{\theta \in \Delta} \neg\theta$.
Then $\psi \in \Psi$ and $\mathcal{A} \models \psi(a)$ contradicting $\mathcal{A} \models \theta(\bar{a})$ for all $\theta \in \Sigma$.]
Let $\mathcal{B}$ and $\bar{b}$ over B be such that $\mathcal{B} \models T \cup \Sigma(\bar{b}) \cup \{\phi(\bar{b})\}$.

Let $\langle \bar{a} \rangle \leq \mathcal{A}, \langle \bar{b} \rangle \leq \mathcal{B}$ be the substructures generated by the entries of $\bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots, b_n)$, respectively.
Since $\bar{a}$ and $\bar{b}$ satisfy exactly the same quantifier free formulas Σ, $b_1 \mapsto a_1, \ldots, \bar{b}_1 \mapsto b_n$ extends to an isomorphism $h \colon \langle \bar{b} \rangle \to \langle \bar{a} \rangle$ (HW).
By assumption, h preserves ϕ, so $\mathcal{A} \models \phi(\bar{a})$.
Hence Claim 1 is proved.

Claim 2: By the Compactness Theorem, Ψ can be replaced by a single quantifier-free formula ψ.

[Let $\bar{c} = (c_1, \ldots, c_n)$ be a tuple of new constants.
Since $T \cup \Psi(\bar{c}) \cup \{\neg\phi(\bar{c})\}$ is not satisfiable, there exists finite $\Delta \subseteq \Psi$ such that $T \cup \Delta(\bar{c}) \cup \{\neg\phi(\bar{c})\}$ is not satisfiable.
Let $\psi(\bar{x})$ be the quantifier-free formula obtained from $\bigwedge \Delta(\bar{c})$ by replacing every occurrence of c_i by x_i back again.
Then $T \models \forall \bar{x}\ \psi(\bar{x}) \to \phi(\bar{x})$.]

Since $\psi \in \Psi$ by definition, $T \models \forall \bar{x}\ \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. $\square$

Theorem

TFAE for a theory T:

1. T has quantifier elimination;
2. For all models $\mathcal{B}, \mathcal{C}$ of T with common substructure $\mathcal{A}$,

$$\mathcal{B}_A \equiv \mathcal{C}_A.$$

3. For all models $\mathcal{B}, \mathcal{C}$ of T with common substructure $\mathcal{A}$ and for all quantifier-free ψ and $\bar{a}$ over A,

$$\mathcal{B} \models \exists y \ \psi(y, \bar{a}) \quad \Rightarrow \quad \mathcal{C} \models \exists y \ \psi(y, \bar{a}).$$

If $\mathcal{L}$ has no constants, $\mathcal{A}$ is allowed to be the empty “structure”.

Proof.

1⇒2. Let ϕ be equivalent mod T to quantifier-free ψ.
Let $\bar{a}$ be a tuple over A.
Note that quantifier-free formulas are preserved under substructures and extensions (HW).

$$\begin{aligned} \mathcal{B} \models \phi(\bar{a}) \quad &\text{iff} \quad \mathcal{B} \models \psi(\bar{a}) \quad \text{iff} \quad \mathcal{A} \models \psi(\bar{a}) \text{ (since } \mathcal{A} \leq \mathcal{B}) \\ &\text{iff} \quad \mathcal{C} \models \psi(\bar{a}) \text{ (since } \mathcal{A} \leq \mathcal{C}) \quad \text{iff} \quad \mathcal{C} \models \psi(\bar{a}). \end{aligned}$$

2⇒3. Clear.

3⇒1. By the previous lemmas it suffices that every $\phi(\bar{x}) = \exists y \; \psi(y, \bar{x})$ is preserved by isomorphisms between substructures of models of T.
Let $\mathcal{B}, \mathcal{C} \models T$, let $\bar{a}$ be a tuple in any $\mathcal{A} \leq \mathcal{B}$, let $h \colon \mathcal{A} \to \mathcal{C}$ be an embedding.
Identifying $\mathcal{A}$ and $h(\mathcal{A})$, assumption 3. yields that $\mathcal{B} \models \phi(\bar{a})$ implies $\mathcal{C} \models \phi(\bar{a})$ (and conversely). □