23. Atomic models

Models realizing only isolated types

A model $\mathcal{A}$ of T is **atomic** if $\mathrm{tp}(\bar{a})$ is isolated for every $\bar{a} \in A^n$.

Note

An atom in a Boolean algebra $\mathcal{B}$ is a minimal non-zero element.
In the Boolean algebra of clopens of a Stone space, atoms are singleton subsets. So isolated types in $S_n(T)$ correspond to atoms.

Theorem

Let T be a complete theory with infinite models over countable $\mathcal{L}$.
Then $\mathcal{A} \models T$ is prime iff $\mathcal{A}$ is countable and atomic.

Proof.

prime $\Rightarrow$ atomic: Let $\mathcal{A}$ be a prime model of T, let $\bar{a} \in A^n$,
let $f \colon \mathcal{A} \to \mathcal{B}$ be an elementary embedding.
Then $f(\bar{a})$ realizes $\mathrm{tp}(\bar{a})$ in $\mathcal{B}$, hence cannot be omitted.
Hence $\mathrm{tp}(\bar{a})$ is isolated by the Omitting Types Theorem.

If $\mathcal{L}$ is countable, T has a countable model and so $\mathcal{A}$ is countable.

For $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ and $C \subseteq A$, call $f\colon C \to B$ a **partial elementary map** if

$$\mathcal{A} \models \phi(\bar{c}) \quad \text{iff} \quad \mathcal{B} \models \phi(f(\bar{c}))$$

for all $\mathcal{L}$-formulas ϕ and all finite sequences $\bar{c}$ from C.

prime $\Leftarrow$ atomic & countable: Let $\mathcal{A} \models T$, countable, atomic.
Let $A = \{a_0, a_1, \dots\}$ and let $\theta_i(\bar{x})$ isolate $\mathrm{tp}^{\mathcal{A}}(a_0, \dots, a_i)$.
Let $\mathcal{B} \models T$. Construct elementary $f\colon \mathcal{A} \to \mathcal{B}$ as union of partial elementary maps

$$f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$$

where $\mathrm{dom} f_i = \{a_0, \dots, a_{i-1}\}$.

- $f_0 := \emptyset$ is elementary since $\mathcal{A} \equiv \mathcal{B}$.

- Given f_s, note that $\mathcal{A} \models \theta_s(a_0, \dots, a_s)$ yields

$$\mathcal{B} \models \exists x \; \theta_s(\underbrace{f_s(a_0)}_{=b_0}, \dots, \underbrace{f_s(a_{s-1})}_{b_{s-1}}, x)$$

Let $b_s \in B$ such that $\mathcal{B} \models \theta_s(b_0, \dots, b_s)$.
Since θ_s isolates $\mathrm{tp}^{\mathcal{A}}(a_0, \dots, a_s)$,

$$\mathrm{tp}^{\mathcal{A}}(a_0, \dots, a_s) = \mathrm{tp}^{\mathcal{B}}(b_0, \dots, b_s).$$

Thus the extension f_{s+1} of f_s by $a_s \mapsto b_s$ is partial elementary. $\square$

Uniqueness of prime models

Theorem

Let T be a complete theory with infinite models over countable $\mathcal{L}$.

1. Any two prime models of T are isomorphic.
2. Let $\bar{a}_1, \bar{a}_2$ be tuples in a prime model $\mathcal{A}$. Then $\mathrm{tp}(\bar{a}_1) = \mathrm{tp}(\bar{a}_2)$ iff $f(\bar{a}_1) = \bar{a}_2$ for some automorphism f of $\mathcal{A}$.

Proof.

Back-and-forth version of the argument in the previous theorem. □

Existence of prime models

Theorem

Let T be a complete theory with infinite models over countable $\mathcal{L}$. Then T has a prime model iff the isolated types in $S_n(T)$ are dense for all $n \in \mathbb{N}$.

Proof.
Note that $\mathcal{A} \models T$ is atomic iff for all $n \in \mathbb{N}$ the partial n-type

$$\Sigma_n := \{\neg\phi(\bar{x}) \mid \phi(\bar{x}) \text{ isolates some type in } S_n(T)\}$$

is not realized in $\mathcal{A}$.
By the Omitting Types Theorem, T has a countable model $\mathcal{A}$ omitting Σ_n for all $n \in \mathbb{N}$ iff Σ_n is not isolated for any $n \in \mathbb{N}$, i.e., for every ψ satisfiable with T there exists $\neg\phi \in \Sigma_n$ such that

$$\begin{aligned} & T \not\models \forall\bar{x}\ \psi(\bar{x}) \to \neg\phi(\bar{x}), \\ \Leftrightarrow\quad & T \cup \{\psi(\bar{x}) \wedge \phi(\bar{x})\} \text{ is satisfiable (since } T \text{ is complete)}, \\ \Leftrightarrow\quad & [\psi] \cap [\phi] \neq \emptyset, \\ \Leftrightarrow\quad & [\psi] \text{ contains the type isolated by } \phi. \end{aligned}$$

Thus T has a countable atomic model iff every non-empty open $[\psi]$ contains some isolated type (i.e. isolated types are dense). □