22. Prime models

Motivation

How far is a structure determined by its language?

Recall

1. A finite structure is uniquely determined (up to isomorphism) by its first order theory.
2. An infinite structure is never determined up to isomorphism by its theory (Löwenheim-Skolem).
3. Some infinite structures are determined by their cardinality and theory (κ-categoricity).

We currently study:

- How do the spaces of types $S_n(T)$ for a complete theory T determine the models of T up to isomorphism?

Prime models

Goal: Study "small" models.

Let T be a complete theory over a countable language $\mathcal{L}$ with infinite models in the following.

A structure $\mathcal{A} \models T$ is a **prime model** of T if $\mathcal{A} \prec \mathcal{B}$ for all $\mathcal{B} \models T$.

Example

$\mathbb{N} := (\mathbb{N}, +, \cdot, <, 0, 1, \}$ is a prime model for $T := \mathrm{Th}(\mathbb{N})$, the true arithmetic.

Let $\mathcal{M} \models T$. Then $\mathbb{N} \to \mathcal{M}$, $n \mapsto \underbrace{1 + \cdots + 1}_{n}$, is an embedding.

We show it is elementary by the Tarski-Vaught Test.

Let $n_1, \ldots, n_k \in \mathbb{N}$ and $\phi(x, y_1, \ldots, y_k)$ be an $\mathcal{L}$-formula such that

$$\mathcal{M} \models \exists x \ \phi(x, n_1, \ldots, n_k)$$

$$\text{i.e.} \quad \mathcal{M} \models \exists x \ \phi(x, \underbrace{1 + \cdots + 1}_{n_1}, \ldots, \underbrace{1 + \cdots + 1}_{n_k}) \quad (\mathcal{L}\text{-sentence})$$

$$\text{i.e.} \quad \mathbb{N} \models \exists x \ \phi(x, \underbrace{1 + \cdots + 1}_{n_1}, \ldots, \underbrace{1 + \cdots + 1}_{n_k})$$

Then for some $m \in \mathbb{N}$ we have

$$\mathbb{N} \models \phi(\underbrace{1 + \cdots + 1}_{m}, \underbrace{1 + \cdots + 1}_{n_1}, \ldots, \underbrace{1 + \cdots + 1}_{n_k})$$

$$\text{i.e.} \quad \mathcal{M} \models \phi(\underbrace{1 + \cdots + 1}_{m}, \underbrace{1 + \cdots + 1}_{n_1}, \ldots, \underbrace{1 + \cdots + 1}_{n_k})$$

$$\text{i.e.} \quad \mathcal{M} \models \phi(m, n_1, \ldots, n_k) \text{ for } m \in \mathbb{N}\,.$$

Thus $\mathbb{N} \prec \mathcal{M}$ by the Tarski-Vaught Test, $\mathbb{N}$ is a prime model of T.

Example

The algebraic closure of $\mathbb{Q}$ is prime for ACF_0. [Proof later.]