21. Omitting types

Recall:

1. A formula ϕ **isolates** a (partial) n-type p of a theory T if

$$T \models \forall \bar{x} \ (\phi(\bar{x}) \rightarrow \psi(\bar{x})) \text{ for all } \psi \in p.$$

2. If T is complete, then every isolated type is realized in any model of T.

Conversely

Theorem (Omitting Types)

Let $\mathcal{L}$ be a countable language, T a satisfiable $\mathcal{L}$-theory and p be a (partial) non-isolated n-type.
Then T has a countable model omitting p.

Proof by Henkin construction (cf. Slides 7).

Let $C := \{c_0, c_1, \dots\}$ be countably many new constant symbols. Inductively construct $T^* = T \cup \{\theta_1, \theta_2, \dots\}$ over $\mathcal{L}_C$ such that

1. T^* has the witness property: for every $\mathcal{L}_C$-formula $\phi(x)$ there exists $c \in C$ such that $\exists x\ \phi(x) \to \phi(c)$ is in T^*;
2. T^* has a model $\mathcal{A}$ with universe $\{c^{\mathcal{A}} \mid c \in C\}$ that omits p: for every $\bar{d} \in C^n$ there exists $\phi \in p$ with $\neg\phi(\bar{d}) \in T^*$.

Let $\phi_0(x), \phi_1(x), \dots$ be an enumeration of $\mathcal{L}_C$-formulas.
Let $\bar{d}_0, \bar{d}_1, \dots,$ be an enumeration of C^n.

We establish 1. and 2. in alternating steps.

Step $2i + 1$: Pick $c \in C$ that does not occur in $T \cup \{\theta_1, \dots, \theta_{2i}\}$ yet and set

$$\theta_{2i+1} := \exists x\ \phi_i(x) \to \phi_i(c).$$

Clearly $T \cup \{\theta_1, \dots, \theta_{2i+1}\}$ is satisfiable.

Step $2i+2$: Let $\bar{d}_i = (c_1, \dots, c_n) \in C^n$.

- Let $\psi(x_1, \dots, x_n)$ be the formula obtained from $\theta_1 \wedge \dots \wedge \theta_{2i+1}$ by replacing each occurrence of c_j by x_j and every other $c \in C \setminus \{c_1, \dots, c_n\}$ by x_c and existentially quantifying over x_c.
- Since ψ does not isolate p, we have $\phi \in p$ such that $T \cup \{\psi, \neg\phi\}$ is satisfiable, i.e., we have an $\mathcal{L}$-model $\mathcal{B}$ of T and $\bar{b} \in B^n$ such that $\mathcal{B} \models \psi(\bar{b}) \wedge \neg\phi(\bar{b})$.
- Set $\theta_{2i+2} := \neg\phi(\bar{d}_i)$.
- $T \cup \{\theta_1, \dots, \theta_{2i+2}\}$ is satisfiable, e.g., by the expansion $\mathcal{B}'$ of $\mathcal{B}$ with $\bar{d}_i^{\mathcal{B}'} := \bar{b}$.

$T^* := T \cup \{\theta_1, \theta_2, \dots\}$ has a model $\mathcal{B}$ by the Compactness Theorem.

- Since T^* has the witness property, $\mathcal{B}$ and $A := \{c^{\mathcal{B}} \mid c \in C\}$ satisfy the assumptions of the Tarski-Vaught Test for elementary substructures (Slides 10).
  [Let $\phi_i(x)$ be an $\mathcal{L}_C$-formula realized in $\mathcal{B}$.
  Since $\mathcal{B} \models \theta_{2i+1}$ we have $\mathcal{B} \models \phi(c^{\mathcal{B}})$ for some $c \in C$.]
- Hence A is the universe of $\mathcal{A} \prec \mathcal{B}$, countable.
- Finally $\mathcal{A}$ omits p since every n-tuple over A is of the form $d_i^{\mathcal{B}}$ and $\mathcal{B} \models \neg\phi(d_i^{\mathcal{B}})$ for some $\phi \in p$ by step $2i+2$. □

Lemma (Tarski-Vaught Test)

Let $\mathcal{B}$ be an $\mathcal{L}$-structure. Then $A \subseteq B$ is the universe of an elementary substructure of $\mathcal{B}$ iff every $\mathcal{L}_A$-formula $\phi(x)$ which is satisfiable in $\mathcal{B}_A$ can be satisfied by an element of A.

Generalizations and restrictions on the Omitting Types Theorem

1. The proof generalizes to omitting countably many non-isolated types (Marker, Thm 4.2.4).
2. Countability of $\mathcal{L}$ is necessary.
   E.g. for $\mathcal{L} = C \cup D$ for disjoint sets of constant symbols C, D with $|C| > \aleph_0, |D| = \aleph_0$, consider

   $$T = \{a \neq b \mid a, b \in C, a \neq b\},$$
   $$p = \{x \neq d \mid d \in D\}.$$

   Since every model of T is uncountable, every model realizes p.

   Any formula $\phi(x)$ contains only finitely many constants from D. Hence if $T \cup \{\phi(x)\}$ is realized in some $\mathcal{A}$ by $a \in A$, we can obtain $\mathcal{A}'$ by redefining some d that does not occur in ϕ as $d^{\mathcal{A}'} := a$ such that $\mathcal{A}' \models T \cup \{\phi(d)\}$. Hence ϕ cannot isolate p.