20. Isolated types

Goal.

The Compactness Theorem yields models realizing types.

How to build models omitting certain types?

Fix an $\mathcal{L}$-theory T.

- Let $\phi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula such that $T \cup \{\phi(\bar{x})\}$ is satisfiable, let p a (partial) n-type for T. Then ϕ **isolates** p if

$$T \models \forall \bar{x} \ (\phi(\bar{x}) \to \psi(\bar{x})) \text{ for all } \psi \in p.$$

- p is **isolated** (**principal**) if it is isolated by some formula.
- Let $S_n(T)$ be the Stone space of all complete n-types of T; basic open sets are $[\phi] := \{p \in S_n(T) \mid \phi \in p\}$.

Lemma

For $p \in S_n(T)$ TFAE:

1. $\{p\}$ is an open subset of $S_n(T)$, i.e. p is an isolated point in the topological sense;
2. $\{p\} = [\phi]$ for some formula ϕ;
3. p is isolated by some formula ϕ.

Proof.

1⇒2. Follows since open sets are unions of basic open sets $[\phi]$ for formulas ϕ.

2⇒3. Assume $\{p\} = [\phi]$.

Let $\mathcal{A}$ be a model for T with $\bar{a} \in A^n$ such that $\mathcal{A} \models \phi(\bar{a})$.

Then $\phi \in \mathrm{tp}^{\mathcal{A}}(\bar{a}) = p$.

Hence $\mathcal{A} \models \psi(\bar{a})$ for all $\psi \in p$, i.e.,

$$T \models \forall \bar{x} \ \big(\phi(\bar{x}) \to \psi(\bar{x})\big).$$

3⇒1. Assume ϕ isolates p, i.e.,

$$T \models \forall \bar{x} \ (\phi(\bar{x}) \to \psi(\bar{x})) \text{ for all } \psi \in p.$$

Then $\neg\phi \notin p$, which implies $\phi \in p$, i.e. $p \in [\phi]$.
Let $q \in [\phi]$. To see $q = p$ note that

- if $\psi \in p$, then every model $\mathcal{A}$ of T and $\bar{a} \in A^n$ realizing ϕ also realizes ψ; hence $\psi \in q$;
- if $\psi \notin p$, then $\neg\psi \in p$ and as above $\psi \notin q$. □

Example: 1-types of dense linear orders

Let $\mathcal{A} := (A, <)$ be a dense linear order without endpoints, let $B \subseteq A$, let $p \in S_1^{\mathcal{A}}(B)$.

Case $x = b$ is in p for some $b \in B$: Then $p = \mathrm{tp}^{\mathcal{A}}(b/B)$ is isolated by $x = b$, i.e., $\{p\} = [x = b]$.

Case $x = b$ is not in p for any $b \in B$: Let

$$L_p := \{b \in B \mid b < x \in p\}, \quad U_p := \{b \in B \mid x < b \in p\}.$$

Then $\ell < u$ for all $\ell \in L_p, u \in U_p$, i.e., L_p, U_P determine a **cut** in $(B, <)$.
[Cleary any cut determines a type.]

Claim: The cut uniquely determines p,

$$\{p\} = \bigcap_{\ell \in L_p} [\ell < x] \ \cap \ \bigcap_{u \in U_p} [x < u].$$

1. Let $p, q \in S_1^{\mathcal{A}}(B)$ with $L_p = L_q$ and $U_p = U_q$.
2. Then p, q contain the same atomic formulas (only $u = v$ or $u < v$), hence the same quantifier-free formulas.

Fact. DLO has quantifier elimination, i.e., every formula ϕ is equivalent to some quantifier-free formula ψ modulo DLO,

$$\mathrm{DLO} \models \forall x \ (\phi(x) \leftrightarrow \psi(x))$$

[Proof later]

3. Hence $p = q$.

Thus 1-types over B can be identified with cuts of B.
Q: Which cuts correspond to isolated points?

Example

$(\mathbb{R}, <)$ realizes all types of $(\mathbb{Q}, <)_{\mathbb{Q}}$.

Omitting types

A model can only omit an isolated type if it omits the isolating formula.

Lemma

1. If ϕ isolates p, then p is realized in any model of $T \cup \{\exists \bar{x}\ \phi(\bar{x})\}$.
2. If T is complete, then every isolated type is realized in any model of T.

Proof.

1. If $\mathcal{A} \models T$ and $\mathcal{A} \models \phi(\bar{a})$, then $\bar{a}$ realizes p.
2. If T is complete and $T \cup \{\phi(\bar{x})\}$ satisfiable, then $T \models \exists \bar{x}\ \phi(\bar{x})$.

$\square$