19. Stone space of complete types

Types with parameters

Recall:

- For an $\mathcal{L}$-structure $\mathcal{A}$ and $B \subseteq A$, let $\mathcal{L}_B$ be obtained from $\mathcal{L}$ by adding a new constant symbol for each $b \in B$.
- $\mathcal{A}_B$ is the $\mathcal{L}_B$-expansion of $\mathcal{A}$ with the natural interpretation of the new symbols.
- An $\mathcal{L}$-formula $\phi(\bar{x}, \bar{b})$, where $\bar{b}$ is a tuple over B, can be viewed as $\mathcal{L}_B$-formula $\phi(\bar{x})$, and conversely.

We extend all previous notions from $\mathcal{L}$ to $\mathcal{L}_B$:

- Types of $\mathcal{A}_B$ are called **types of $\mathcal{A}$ over B**.
- $\mathrm{tp}^{\mathcal{A}}(\bar{a}/B) := \{\mathcal{L}_B\text{-formulas } \phi(\bar{x}) \ : \ \mathcal{A}_B \models \phi(\bar{a})\}$ is the **complete type of $\bar{a}$ over B**.
- $S_n^{\mathcal{A}}(B)$ is the set of all complete n-types of $\mathcal{A}$ over B.

Example: $\mathbb{Q} := (\mathbb{Q}, <)$ and $B := \mathbb{N}$

1. $p(x) := \{x > 1, x > 2, x > 3, \dots\}$ is a partial 1-type of $\mathbb{Q}$ over $\mathbb{N}$, but omitted by $\mathbb{Q}$.
2. $\mathrm{tp}^{\mathbb{Q}}(\frac{1}{2}/\,\mathbb{N})$ contains e.g. $x < 1$ but not $x < 0$.
3. Let $r \in \mathbb{Q}$ with $0 < r < 1$. By homogeneity, there exists $a \in \mathrm{Aut}\,(\mathbb{Q}, <)$ that fixes every $n \in \mathbb{N}$ and $a(\frac{1}{2}) = r$.
4. Since a is an $\mathcal{L}_B$-automorphism, every $\mathcal{L}_B$-formula ϕ satisfies

$$\mathbb{Q} \models \phi(\frac{1}{2}) \quad \text{iff} \quad \mathbb{Q} \models \phi(r).$$

5. Hence r realizes $\mathrm{tp}^{\mathbb{Q}}(\frac{1}{2}/\,\mathbb{N})$.
6. $\mathrm{tp}^{\mathbb{Q}}(r/\,\mathbb{N}) = \mathrm{tp}^{\mathbb{Q}}(\frac{1}{2}/\,\mathbb{N})$ iff $0 < r < 1$.

Complete types express what possible first order properties elements have in elementary extensions

Corollary

$p \in S_n^{\mathcal{A}}(B)$ iff there exist an elementary extension $\mathcal{C}$ of $\mathcal{A}$ and $\bar{c} \in C^n$ such that $p = \mathrm{tp}^{\mathcal{C}}(c/B)$.

Proof.

$\Leftarrow$: Assume $\mathcal{A} \prec \mathcal{C}$.

- Then $\mathrm{Th}(\mathcal{C}_B) = \mathrm{Th}(\mathcal{A}_B)$ yields $S_n^{\mathcal{C}}(B) = S_n^{\mathcal{A}}(B)$.
- So for $\bar{c} \in C^n$ we have $\mathrm{tp}^{\mathcal{C}}(\bar{c}/B) \in S_n^{\mathcal{C}}(B) = S_n^{\mathcal{A}}(B)$.

$\Rightarrow$: Let $p \in S_n^{\mathcal{A}}(B)$.

- By the Lemma (Realizing Types), there exist an elementary extension $\mathcal{C}$ of $\mathcal{A}$ and $\bar{c} \in C^n$ realizing p.
- Then $p \subseteq \mathrm{tp}^{\mathcal{C}}(\bar{c}/B)$ and equality follows since p is complete. $\square$

Stone spaces

There is a natural topology on the space of complete n-types $S_n^{\mathcal{A}}(B)$ called the **Stone topology**:

- The basic open sets are

$$[\phi] := \{p \in S_n^{\mathcal{A}}(B) \mid \phi \in p\}$$

  for ϕ an $\mathcal{L}_B$-formula with free variables $x_1, \dots, x_n$
  (cf. compact topological space of complete $\mathcal{L}$-theories).
- For complete p, either $\phi \in p$ or $\neg\phi \in p$. So $[\phi] = S_n^{\mathcal{A}}(B) \setminus [\neg\phi]$ is closed and open (**clopen**).
- The clopen sets in $S_n^{\mathcal{A}}(B)$ form a **Boolean algebra** with $[\phi] \cup [\psi] = [\phi \vee \psi]$ and $[\phi] \cap [\psi] = [\phi \wedge \psi]$.
  (Complete types are the ultrafilters of that Boolean algebra.)

A **Stone space** is a non-empty topological space X that is

- **compact** (i.e., every open cover of X has a finite subcover),
- **totally separated** (i.e., for any distinct $x, y \in X$ there exist disjoint open sets U, V such that $x \in U, y \in V$ and $U \cup V = X$).

(cf. Stone Representation Theorem for Boolean algebras)

Lemma

$S_n^{\mathcal{A}}(B)$ is a Stone space.

Proof.

totally separated: For distinct $p, q \in S_n^{\mathcal{A}}(B)$, there exists $\phi \in p$ and $\phi \notin q$, hence $\neg\phi \in q$. Then $p \in [\phi]$ and $q \in [\neg\phi]$.

compact: Let $O := \{[\phi_i] \mid i \in I\}$ be a cover of $S_n^{\mathcal{A}}(B)$.

- Let $\mathcal{A} \prec \mathcal{C}$ and $\bar{c} \in C^n$. Then $\mathrm{tp}^{\mathcal{C}}(\bar{c}/B)$ is a complete type of $\mathcal{A}$ over B, hence in $S_n^{\mathcal{A}}(B)$ by the Corollary above.
- Hence $\mathcal{C}_B \models \phi_i(\bar{c})$ for some $i \in I$.
- Thus $\mathrm{Th}(\mathcal{A}_B) \cup \{\neg\phi_i \mid i \in I\}$ is not satisfiable (in any elementary extension $\mathcal{C}$ of $\mathcal{A}$).
- By Compactness, we have a finite $F \subseteq I$ such that $\mathrm{Th}(\mathcal{A}_B) \cup \{\neg\phi_i \mid i \in F\}$ is not satisfiable.
- So no type $p \in S_n^{\mathcal{A}}(B)$ contains $\{\neg\phi_i \mid i \in F\}$; by completeness, every p contains ϕ_i for some $i \in F$.
- Thus $\{[\phi_i] \mid i \in F\}$ is a finite subcover of O. $\square$