18. Types

Fix a language $\mathcal{L}$ for all formulas, theories, structures throughout.

Intuition

Types are sets of formulas expressing properties of elements from models of a theory T.

Example

For the theory of $(\mathbb{N}, +, \cdot, 0, 1, <)$,

$$\{1 < x, 1 + 1 < x, 1 + 1 + 1 < x, \ldots\}$$

is satisfied by ∞, which does not exist in $\mathbb{N}$ but in some non-standard model of arithmetic.

Satisfiable formulas

- A set Σ of formulas with free variables $x_1, \ldots, x_n$ is **satisfiable over a structure** $\mathcal{A}$ (or **realized in** $\mathcal{A}$) if there exist $a_1, \ldots, a_n \in A$ such that

$$\mathcal{A} \models \phi(a_1, \ldots, a_n) \quad \text{for all } \phi \in \Sigma.$$

- Σ is **satisfiable** if it is satisfiable over some $\mathcal{A}$.

Lemma

A set Σ of formulas with free variables $x_1, \ldots, x_n$ is satisfiable iff all finite subsets of Σ are satisfiable.

Proof.

1. Add new constant symbols $c_1, \ldots, c_n$ to the language.
2. Σ is satisfiable iff the set of sentences $\{\phi(c_1, \ldots, c_n) \mid \phi(x_1, \ldots, x_n) \in \Sigma\}$ is satisfiable.
3. Use the Compactness Theorem on the latter. □

n-types of a theory

- For $n \in \mathbb{N}$, a **(partial) n-type of a theory** T is a set p of formulas with free variables $x_1, \ldots, x_n$ such that $p \cup T$ is satisfiable.
- An **n-type of a structure** $\mathcal{A}$ is an n-type of $\mathrm{Th}(\mathcal{A})$.
- $\mathcal{A}$ **omits** a type p if p is not realized in $\mathcal{A}$.

Example

For $\mathbb{N} := (\mathbb{N}, +, \cdot, 0, 1, <)$

$$\Sigma := \{1 < x, 1 + 1 < x, 1 + 1 + 1 < x, \ldots\}$$

is a (partial) 1-type of $\mathrm{Th}(\mathbb{N})$ but $\mathbb{N}$ omits Σ.

Lemma (Realizing Types)

For a set Σ of $\mathcal{L}$-formulas with free variables $x_1, \dots, x_n$ and an $\mathcal{L}$-structure $\mathcal{A}$ TFAE:

1. Σ is an n-type of $\mathcal{A}$;
2. Every finite subset of Σ is realized in $\mathcal{A}$;
3. Σ is realized in some elementary extension of $\mathcal{A}$.

Proof.

1⇒2. Write $\bar{x} := (x_1, \dots, x_n)$, let $\Sigma(\bar{x})$ be an n-type of $\mathcal{A}$.
Then we have a model $\mathcal{B}$ of $\mathrm{Th}(\mathcal{A})$ and $\bar{b} \in B^n$ such that $\mathcal{B} \models \Sigma(\bar{b})$.
For finite $\Delta \subseteq \Sigma$, we have $\mathcal{B} \models \exists \bar{x} \ \bigwedge \Delta(\bar{x})$.
Since $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$, we also have $\mathcal{A} \models \exists \bar{x} \ \bigwedge \Delta(\bar{x})$.
Hence Δ is realized in $\mathcal{A}$.

2⇒3. Assume every finite $\Delta \subseteq \Sigma$ is realized in $\mathcal{A}$.
Then $\mathrm{Th}(\mathcal{A}_A) \cup \Sigma$ is finitely satisfiable, hence has a model $\mathcal{B}$ by the Compactness Theorem.
The $\mathcal{L}$-reduct of $\mathcal{B}$ is an elementary extension of $\mathcal{A}$ which realizes Σ.

3⇒1. clear. □

Complete types

- An n-type p is **complete** if for every formula $\phi(x_1, \ldots, x_n)$ either $\phi \in p$ or $\neg\phi \in p$.
- For a structure $\mathcal{A}$ and an n-tuple $\bar{a}$ over A,

$$\mathrm{tp}^{\mathcal{A}}(\bar{a}) := \{\phi(\bar{x}) \mid \mathcal{A} \models \phi(\bar{a})\}$$

  is the **(complete) type** of $\bar{a}$.

Example

By homogeneity, $\mathbb{Q} := (\mathbb{Q}, <)$ has precisely

- one complete 1-type $\mathrm{tp}^{\mathbb{Q}}(0)$.
- What are the complete 2-types?