16. Fraïssé's theorem

Recall: The age of a homogeneous structure has the HP, JEP, AP.

Let $\mathcal{L}$ be countable. A non-empty class K of finitely generated $\mathcal{L}$-structures is a **Fraïssé class** (an **amalgamation class**) if K has the HP, JEP, AP, is closed under isomorphisms and contains at most countably many structures up to isomorphism.

Theorem (Fraïssé)

Let K be a Fraïssé class of $\mathcal{L}$-structures. Then there exists a homogeneous, at most countable $\mathcal{L}$-structure $\mathcal{C}$ with $\mathrm{Age}(\mathcal{C}) = K$. This $\mathcal{C}$ is unique up to isomorphism (called the **Fraïssé limit** of K).

Example

- $(\mathbb{Q}, <)$ is the Fraïssé limit of the class of finite linear orders.
- The $\mathbb{Q}$-vector space of countable dimension is the Fraïssé limit of the class of fin dim $\mathbb{Q}$-vector spaces.
- The class of finite partially ordered sets is Fraïssé with Fraïssé limit the **homogeneous universal partial order**.
- The class of finite fields of prime characteristic p is Fraïssé with Fraïssé limit the algebraic closure of the prime field F_p.
- The class of finite groups is Fraïssé with **Hall's universal group** as Fraïssé limit.
- The class of finite graphs has Fraïssé limit the **countable random graph** (**Rado graph**).

More random facts about the random graph

1. The random graph embeds every finite or countable infinite graph.
2. (Erdös, Renyi) The random graph is a countably infinite graph that can be constructed (with probability one) by choosing independently at random for each pair of its vertices whether to connect the vertices by an edge or not.
3. (Ackerman, Rado) The random graph on vertex set $\mathbb{N}$ has an edge between x and y for $x < y$ if the xth bit of y in binary is nonzero.
4. The random graph satisfies the **extension property**: for any disjoint finite sets of vertices U and V there exists a vertex x not in $U \cup V$ which is adjacent to every element in U and to none in V.
5. For any partition of the random graph in finitely many subsets, at least one of the induced subgraphs is isomorphic to the random graph.

Uniqueness proof of Fraïssé's Theorem

Let $\mathcal{C}, \mathcal{D}$ be countable homogeneous of the same age,
with enumerations $C = \{c_0, c_1, \dots\}$, $D = \{d_0, d_1, \dots\}$.
We use a back-and-forth argument.

Assume $f \colon \mathcal{C} \to \mathcal{D}$ is already defined on a finitely generated $\mathcal{A} \leq \mathcal{C}$.

Going forth

1. Let $i \in \mathbb{N}$ be smallest such that $c_i \notin A$.
2. Let $\mathcal{B} \leq \mathcal{C}$ be generated by $A \cup \{c_i\}$. Then $\mathcal{B}$ is fin generated.
3. Since $\mathrm{Age}(\mathcal{C}) = \mathrm{Age}(\mathcal{D})$, we have $e \colon \mathcal{B} \hookrightarrow \mathcal{D}$.
4. By the homogeneity of $\mathcal{D}$, the partial isomorphism $fe^{-1} \colon e(\mathcal{A}) \to f(\mathcal{A})$ extends to an automorphism a of $\mathcal{D}$.
5. Then the embedding $\mathcal{B} \to \mathcal{D},\ x \mapsto ae(x)$, extends f from the domain A to B.

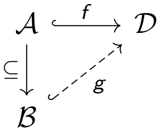
Going back (swap the roles for $\mathcal{C}$ and $\mathcal{D}$)

1. Let $i \in \mathbb{N}$ be smallest such that $d_i \notin f(A)$.
2. Consider the substructure of $\mathcal{D}$ generated by $f(A) \cup \{d_i\}$ and find a copy of it in $\mathcal{C}$ to extend f with d_i in the image of the extension.

Alternating the back-and-forth steps, we grow an isomorphism $f : \mathcal{C} \to \mathcal{D}$.
The uniqueness part of Fraïssé's Theorem is proved. □

Weak homogeneity

$\mathcal{D}$ is **weakly homogeneous** if for all $\mathcal{A}, \mathcal{B} \in \mathrm{Age}(\mathcal{D})$ with $\mathcal{A} \leq \mathcal{B}$ and $f\colon \mathcal{A} \to \mathcal{D}$ there exists $g\colon \mathcal{B} \to \mathcal{D}$ that extends f.



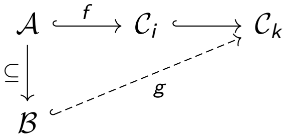
Note

- In the previous proof, weak homogeneity of $\mathcal{C}$ and $\mathcal{D}$ suffices to construct the isomorphism $f\colon \mathcal{C} \to \mathcal{D}$.
- For $\mathcal{C} = \mathcal{D}$ weakly homogeneous, any isomorphism between fin generated substructures extends to an automorphism of $\mathcal{C}$. Hence weakly homogeneous is equivalent to homogeneous.
- In practice weak homogeneity is easier to check.

Existence proof

Construct $\mathcal{C}$ as a union of a chain $\mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \dots$ of structures in K such that

(∗) for all $\mathcal{A} \leq \mathcal{B}$ in K and $f : \mathcal{A} \hookrightarrow \mathcal{C}_i$ for some i, there exist $k \geq i$ and $g : \mathcal{B} \hookrightarrow \mathcal{C}_k$ extending f.



Then

- $\mathcal{C} := \lim_{i \in \mathbb{N}} \mathcal{C}_i$ is (weakly) homogeneous by (∗).
- $\mathrm{Age}(\mathcal{C}) = K$.
  [⊆ is clear. For ⊇, note that for each $\mathcal{A} \in K$ there exists $\mathcal{B} \in K$ which embeds both $\mathcal{A}$ and $\mathcal{C}_0$ by JEP. Then $\mathcal{B}$ embeds into some $\mathcal{C}_k$ by (∗). Hence $\mathcal{A} \in \mathrm{Age}(\mathcal{C})$.]

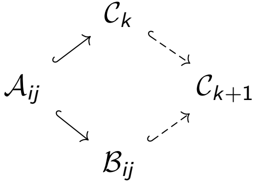
Construct $\mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \ldots$ satisfying $(*)$ inductively.

Fix some $\mathcal{C}_0 \in \mathcal{K}$.

Let P be a countable set of representatives of $\{(\mathcal{A}, \mathcal{B}) \in K^2 \mid \mathcal{A} \leq \mathcal{B}\}$ up to isomorphism. Let $\pi \colon \mathbb{N}^2 \to \mathbb{N}$ be a bijection such that $\pi(i,j) \geq i$ for all $i, j \in \mathbb{N}$.

Assume $\mathcal{C}_k$ is constructed. Fix a list $(f_{kj}, \mathcal{A}_{kj}, \mathcal{B}_{kj})_{j \in \mathbb{N}}$ of all triples $(f, \mathcal{A}, \mathcal{B})$ such that $(\mathcal{A}, \mathcal{B}) \in P$ and $f \colon \mathcal{A} \hookrightarrow \mathcal{C}_k$.

For $i, j \in \mathbb{N}$ such that $k = \pi(i,j)$, construct $\mathcal{C}_{k+1}$ as amalgam,



so that f_{ij} extends to an embedding $\mathcal{B}_{ij} \hookrightarrow \mathcal{C}_{k+1}$.
Then $(*)$ holds for all $(f_{ij}, \mathcal{A}_{ij}, \mathcal{B}_{ij})$.

The existence part of Fraïssé's Theorem is proved. □

Outlook

A Fraïsse class K is **uniformly locally finite** if there exists a function $f: \mathbb{N} \to \mathbb{N}$ such that every n-generated structure in K has size $\leq f(n)$.

K is clearly uniformly locally finite if its language is finite and has no function symbols.

Theorem

Let $\mathcal{C}$ be the Fraïssé limit of a uniformly locally finite Fraïsse class. Then $\mathrm{Th}(\mathcal{C})$ is $\aleph_0$-categorical.