15. Homogeneous structures

Goal

Construct countable structures as limit of their finite substructures by **Fraïssé amalgamation**.

The age of a structure

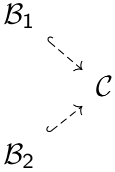
The **age** of an $\mathcal{L}$-structure $\mathcal{A}$ is the class $\mathrm{Age}(\mathcal{A})$ of all finitely generated $\mathcal{L}$-structures that embed into $\mathcal{A}$.

Example

- $\mathrm{Age}(\mathbb{Q}, <)$
- Age of the $\mathbb{Q}$-vector space of countable dimension

If K is the age of some structure, then $K \neq \emptyset$ and K satisfies the

1. **Hereditary property (HP)**: If $\mathcal{B} \in K$ and $\mathcal{C} \leq \mathcal{B}$ is finitely generated, then $\mathcal{C} \in K$.
2. **Joint embedding property (JEP)**: For all $\mathcal{B}_1, \mathcal{B}_2 \in K$ there exists $\mathcal{C} \in K$ that embeds both $\mathcal{B}_1$ and $\mathcal{B}_2$.



Lemma (Fraïssé)

For a non-empty class K of finitely generated $\mathcal{L}$-structures TFAE:

1. K is the age of a finite or countable $\mathcal{L}$-structure;
2. K has the HP and JEP, is closed under isomorphisms and contains at most countably many structures up to isomorphism.

Proof.

1.⇒2. Easy.

2.⇒1. Enumerate representatives $\mathcal{A}_0, \mathcal{A}_1, \ldots$ for the isomorphism classes in K.

Define a chain $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1 \hookrightarrow \ldots$ as follows:

- $\mathcal{B}_0 := \mathcal{A}_0$
- Given $\mathcal{B}_i$, find $\mathcal{B}_{i+1} \in K$ such that $\mathcal{B}_i \hookrightarrow \mathcal{B}_{i+1}$ and $\mathcal{A}_{i+1} \hookrightarrow \mathcal{B}_{i+1}$ by the JEP.

Set $\mathcal{B} := \lim_{i \in \mathbb{N}} \mathcal{B}_i$.

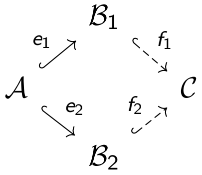
- $\mathcal{B}$ is at most countable as the union of countably many at most countable structures.
- Every structure in K embeds into $\mathcal{B}$ by construction.
- For any finitely generated substructure $\mathcal{A}$ of $\mathcal{B}$, the finitely many generators lie in some $\mathcal{B}_i$. So $\mathcal{A} \hookrightarrow \mathcal{B}_i$ and $\mathcal{A} \in K$ by the HP.
- Thus $\mathrm{Age}(\mathcal{B}) = K$. $\square$

Amalgamation

Note: all infinite linear orders have the same age (the class of finite linear orders).
In which sense do fin linear orders tend to $(\mathbb{Q}, <)$ instead of $(\mathbb{Z}, <)$?

A class K of $\mathcal{L}$-structures has the **amalgamation property (AP)** if for all $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in K$ and embeddings $e_1 \colon \mathcal{A} \hookrightarrow \mathcal{B}_1$, $e_2 \colon \mathcal{A} \hookrightarrow \mathcal{B}_2$ there exist $\mathcal{C} \in K$ and embeddings $f_1 \colon \mathcal{B}_1 \hookrightarrow \mathcal{C}$, $f_2 \colon \mathcal{B}_2 \hookrightarrow \mathcal{C}$ such that

$$f_1 e_1 = f_2 e_2.$$



Example

The following classes have AP: finite linear orders, finite graphs, finite dimensional F-vector spaces.

Homogeneity

A structure $\mathcal{A}$ is **(ultra)homogeneous** if every isomorphism between finitely generated substructures of $\mathcal{A}$ extends to an automorphism of $\mathcal{A}$.

Homogeneity can often be proved via the back-and-forth method.

Example

$(\mathbb{Q}, <)$ is homogeneous.

- Let f_0 be any isomorphism between two finite substructures.
- Replacing the empty function by f_0 in the back-and-forth argument for the $\aleph_0$-categoricity of DLO, f_0 can be extended to an automorphism of $(\mathbb{Q}, <)$.

$(\mathbb{Z}, <)$ is not homogeneous (Why?).

Example

Vector spaces are homogeneous (assuming AC).

Lemma

The age of any homogeneous structure $\mathcal{B}$ has the AP.

Proof.

1. Let $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in \mathrm{Age}(\mathcal{B})$ with embeddings $e_i \colon \mathcal{A} \hookrightarrow \mathcal{B}_i$ for $i = 1, 2$.
2. Let $g_i \colon \mathcal{B}_i \hookrightarrow \mathcal{B}$.
3. $g_2 e_2 e_1^{-1} g_1^{-1}$ restricts to an isomorphism $g_1 e_1(\mathcal{A}) \to g_1 e_1(\mathcal{A})$, which extends to an automorphism a of $\mathcal{B}$ by homogeneity.
4. Let $\mathcal{C} \leq \mathcal{B}$ be generated by $ag_1(\mathcal{B}_1) \cup g_2(\mathcal{B}_2)$.
   Then $\mathcal{C}$ is finitely generated and hence in $\mathrm{Age}(\mathcal{B})$.
5. $ag_1 \colon \mathcal{B}_1 \to \mathcal{B}$ and $g_2 \colon \mathcal{B}_2 \to \mathcal{B}$ are embeddings into $\mathcal{C}$ such that
   $$ag_1 e_1(x) = g_2 e_2(x) \text{ for all } x \in A.$$
6. Hence $\mathrm{Age}(\mathcal{A})$ has the AP. □

Example

Since $(\mathbb{Q}, <)$ is homogeneous, its age (the class of all finite linear orders) has the AP.