14. Back-and-forth arguments

DLO

DLO is the theory of **dense linear orders without endpoints** in the language $\mathcal{L} = \{<\}$ axiomatized by the axioms for partial orders (reflexivity, anti-symmetry, transitivity) as well as

- $\forall x \forall y \; x < y \lor x > y \lor x = y$ (linearity)
- $\forall x \forall y \; (x < y \to \exists z \; x < z < y)$ (density)
- $\forall x \exists y \exists z \; y < x < z$ (no end points)

Example

$(\mathbb{Q}, <) \models$ DLO

Theorem (Cantor)

DLO is $\aleph_0$-categorical and complete.

Back-and-forth method

Construct an isomorphism f between countable structures $\mathcal{A}$ and $\mathcal{B}$ iteratively from a sequence of **partial isomorphisms** $f_i\colon \mathcal{A}_i \to \mathcal{B}_i$ between finite substructures

$$\mathcal{A}_0 \leq \mathcal{A}_1 \leq \mathcal{A}_2 \leq \cdots \leq \mathcal{A} \quad \text{with} \bigcup_{i\in\mathbb{N}} \mathcal{A}_i = \mathcal{A}$$

$$\mathcal{B}_0 \leq \mathcal{B}_1 \leq \mathcal{B}_2 \leq \cdots \leq \mathcal{B} \quad \text{with} \bigcup_{i\in\mathbb{N}} \mathcal{B}_i = \mathcal{B}$$

Then $f := \bigcup_{i\in\mathbb{N}} f_i$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

f_i are built using two alternating steps:

- (**forth**) extend the domain to guarantee $\bigcup \mathrm{dom} f_i = A$,
- (**back**) extend the range to guarantee $\bigcup \mathrm{im} f_i = B$.

Proof.

Let $\mathcal{A} := (A, <)$ and $\mathcal{B} := (B, <)$ be countable models of DLO with enumerations $A = \{a_0, a_1, \dots\}$ and $B = \{b_0, b_1, \dots\}$.

Set $A_0 := B_0 := \emptyset$ and f_0 the empty function.
Suppose $f_i \colon A_i \to B_i$ is already defined.

Going forth

1. Let $j \in \mathbb{N}$ be smallest such that $a_i \notin A_i$.
2. Let a_- be greatest in A_i such that $a_- < a_j$ (or $a_- := -\infty$ if no such element exists).
3. Let a_+ be smallest in A_i such that $a_+ > a_j$ (or $a_+ := \infty$ if no such element exists).
4. Let $b \in B$ such that $f_i(a_-) < b < f_i(a_+)$ where $f_i(-\infty) = -\infty$, $f_i(\infty) = \infty$.
   Such b exists since $<$ on B is dense and has no end points.
5. Set $A_i' := A_i \cup \{a_j\}$, $B_i' := B_i \cup \{b\}$ and f_i' the extension of f_i by $a_j \mapsto b$.
6. Then f_i' is an order isomorphism.

Next suppose $f'_i \colon A'_i \to B'_i$ is already defined.
Going back (exchange the roles for $\mathcal{A}$ and $\mathcal{B}$)

1. Let $j \in \mathbb{N}$ be smallest such that $b_j \notin B'_i$.
2. As above find $a \in A$ such that the map f_{i+1} extending f'_i by $a \mapsto b_j$ is an isomorphism from $A_{i+1} := A'_i \cup \{a\}$ to $B_{i+1} := B'_i \cup \{b_j\}$.

Alternating the forth and back steps yields a sequence

$$f_0 \subseteq f_1 \subseteq f_2 \ldots$$

such that $f := \bigcup_{i \in \mathbb{N}} f_i$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.
Thus DLO is $\aleph_0$-categorical. □

Homogeneity

A structure $\mathcal{A}$ is **(ultra)homogeneous** if every isomorphism between finitely generated substructures of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$.

Homogeneity can often be proved via the back-and-forth method.

Example

$(\mathbb{Q}, <)$ is homogeneous.

- Let f_0 be any isomorphism between two finite substructures.
- Replacing the empty function by f_0 in the above back-and-forth argument, f_0 can be extended to an automorphism of $(\mathbb{Q}, <)$.

Example

Vector spaces are homogeneous.

Outlook

Countable homogeneous (and $\aleph_0$-categorical) structures can be constructed from their class of finite substructures by **Fraïssé amalgamation**.