13. Algebraically closed fields

Comparing ACF_0 and ACF_p for p prime

Corollary (Transfer Principle for algebraically closed fields)

For a sentence ϕ in the language of rings TFAE:

1. $\mathbb{C} \models \phi$.
2. Every algebraically closed field of characteristic 0 satisfies ϕ.
3. Some algebraically closed field of characteristic 0 satisfies ϕ.
4. For every $m \in \mathbb{N}$ there exists a prime $p > m$ such that some alg closed field of char p satisfies ϕ.
5. There exists $m \in \mathbb{N}$ such that for all primes $p > m$ all alg closed fields of char p satisfy ϕ.

Proof.

1. iff 2. iff 3. is completeness of ACF_0.

5. $\Rightarrow$ 4. is clear.

2. $\Rightarrow$ 5.

- Assume $\mathrm{ACF}_0 \models \phi$.
- By the Compactness Theorem, we have a finite $\Delta \subseteq \mathrm{ACF}_0$ such that $\Delta \models \phi$.
- Since Δ contains only finitely many sentences $\underbrace{1 + \cdots + 1}_{n} \neq 0$, for p large enough, $\mathrm{ACF}_p \models \Delta$.
- Hence $\mathrm{ACF}_p \models \phi$ for all large enough p.

4. $\Rightarrow$ 2. by contraposition.

- Assume $\mathrm{ACF}_0 \not\models \phi$, i.e., $\mathrm{ACF}_0 \models \neg\phi$ by completeness.
- As above, $\mathrm{ACF}_p \models \neg\phi$ for all large enough p, i.e., 4. does not hold.

$\square$

Theorem (Ax-Grothendieck, special case)

Every injective map

$$\mathbb{C}^n \to \mathbb{C}^n,\ \bar{x} \mapsto (f_1(\bar{x}), \ldots, f_n(\bar{x})),$$

for polynomials $f_1, \ldots, f_n$ is bijective.

($\mathbb{C}^n$ can be replaced by an algebraic variety over an alg closed field.)

Proof sketch.

1. For F a finite field instead of $\mathbb{C}$, the statement is clear.
2. Since the algebraic closure $\overline{F}_p$ of the p-element field F_p is a union of finite fields, the statement follows for $\overline{F}_p$.
3. There is a sentence $\phi_{n,d}$ such that for any field F we have $F \models \phi_{n,d}$ iff every injective polynomial function on F^n whose components f_i have degree $\leq d$ is surjective.
4. Since $\overline{F}_p \models \phi_{n,d}$ for all p, $\mathbb{C} \models \phi_{n,d}$ by the previous Corollary.

□