12. κ-categorical structures

Categoricity

By the Löwenheim-Skolem Theorem,
no theory with infinite model can define this model up to isomorphism.
The best we can hope for is a unique model for a given cardinality.

Let κ be an infinite cardinal. A theory is κ-**categorical** if it has exactly one model of cardinality κ up to isomorphism.

Example

1. The theory of F-vector spaces is κ-categorical for all $\kappa > |F|$.
2. The theory of torsion-free divisible Abelian groups is κ-categorical for all $\kappa > \aleph_0$.
3. The theory of algebraically closed fields ACF_p of characteristic p is κ-categorical for all $\kappa > \aleph_0$.

Theorem (Vaught's test)

Let T be an $\mathcal{L}$-theory that is

- satisfiable,
- κ-categorical for some infinite $\kappa \geq |\mathcal{L}|$ and
- has no finite models.

Then T is complete.

Example

1. The following are complete: the theory of infinite vector spaces over a fixed field F, ACF_p
2. The theory of all F-vector spaces is not complete (Why?)

Outlook

- κ-categoricity is a very strong property of (complete) theories.
- $\aleph_0$-categorical theories have very special properties distinct from κ-categorical theories for uncountable κ.

Morley's Categoricity Theorem

If a complete theory T in a countable language is κ-categorical for some uncountable κ, then T is κ-categorical for all uncountable κ.

Hence, if a complete theory in a countable language with an infinite model is κ-categorical, then either for

- all infinite κ,
- all uncountable κ,
- just $\kappa = \aleph_0$.