11. Löwenheim-Skolem Theorem

Limits of chains of structures

As a consequence of the Compactness Theorem,
a first order theory cannot specify the cardinality of an infinite model.
More precisely:

Theorem (Löwenheim-Skolem)

Let $\mathcal{A}$ be an infinite $\mathcal{L}$-structure, $S \subseteq A$ and κ an infinite cardinal.

- (Downwards) If $\max(|S|, |\mathcal{L}|) \leq \kappa \leq |A|$, then $\mathcal{A}$ has an elementary substructure containing S of cardinality κ.
- (Upwards) If $\max(|A|, |\mathcal{L}|) \leq \kappa$, then $\mathcal{A}$ has an elementary extension of cardinality κ.

Proof.

Downwards: Choose $S \subseteq S' \subseteq A$ with $|S'| = \kappa$ and apply the Corollary of the Tarski-Vaught test.

Upwards: First construct an elementary extension $\mathcal{C}$ of $\mathcal{A}$ such that $|C| \geq \kappa$.

1. Let D be a set of new constants, $|D| = \kappa$.
2. Since $\mathcal{A}$ is infinite,

$$T := \mathrm{Th}(\mathcal{A}_A) \cup \{\neg(c = d) \mid c, d \in D, c \neq d\}$$

   is finitely satisfiable (even by expansions of $\mathcal{A}_A$).
3. By the Compactness Theorem, T has a model $\mathcal{C}'_A$ of size $\geq \kappa$.
4. Its $\mathcal{L}$-reduct $\mathcal{C}$ is an elementary extension of $\mathcal{A}$.

Finally apply the downward part to $\mathcal{C}$ and $S = A$ to get an elementary substructure $\mathcal{B}$ of $\mathcal{C}$ with $|B| = \kappa$. Then $\mathcal{B}$ is an elementary extension of $\mathcal{A}$ (HW). □

Corollary

If an $\mathcal{L}$-theory has some infinite model, then it has a model of every infinite cardinality $\geq |\mathcal{L}|$.