10. Elementary substructures

Limits of chains of structures

1. Let I be linearly ordered by $\leq$.
2. A sequence of $\mathcal{L}$-structures $(\mathcal{A}_i)_{i\in I}$ is a **chain** if $\mathcal{A}_i$ is a substructure of $\mathcal{A}_j$ for all $i \leq j$.
3. The **union** of a chain $(\mathcal{A}_i)_{i\in I}$ is the $\mathcal{L}$-structure $\mathcal{B} = \lim_{i\in I} A_i$ such that

- $B := \bigcup_{i\in I} A_i$,
- $R^{\mathcal{B}} := \bigcup_{i\in I} R^{\mathcal{A}_i}$ for any relation symbol $R \in \mathcal{L}$,
- $c^{\mathcal{B}} := c^{\mathcal{A}_i}$ for some (hence all) $i \in I$ for any constant $c \in \mathcal{L}$,
- $f^{\mathcal{B}}(\bar{a}) := f^{\mathcal{A}_i}(\bar{a})$ if $\bar{a}$ is a tuple in A_i for some $i \in I$ for any function symbol $f \in \mathcal{L}$.

Example

For $n \in \mathbb{N}$ let

$$\mathcal{A}_n := (\{-n, -n+1, \dots\}, \leq).$$

1. $\lim_{n\in\mathbb{N}} \mathcal{A}_n =$ $(\mathbb{Z}, \leq)$
2. $\mathrm{Th}(A_n) = \mathrm{Th}(A_0)$ since $\mathcal{A}_n \cong \mathcal{A}_0$
3. $\mathrm{Th}(A_n) \neq \mathrm{Th}(\lim_{n\in\mathbb{N}} \mathcal{A}_n)$ because $\mathcal{A}_n$ has a smallest element but $\mathbb{Z}$ not. $\exists x \forall y \quad x \leq y$

Elementary embeddings

Recall that $\mathcal{L}$-embeddings preserve and reflect relations, functions, constants in $\mathcal{L}$.
An $\mathcal{L}$-embedding $h\colon \mathcal{A} \to \mathcal{B}$ is an **elementary embedding** if

$$\mathcal{A} \models \phi(a_1, \ldots, a_n) \text{ iff } \mathcal{B} \models \phi(h(a_1), \ldots, h(a_n))$$

for all $\mathcal{L}$-formulas ϕ and $a_1, \ldots, a_n \in A$.
A substructure $\mathcal{A}$ of $\mathcal{B}$ is an **elementary substructure** (written $\mathcal{A} \prec \mathcal{B}$) if the inclusion map is elementary. Then $\mathcal{B}$ is an **elementary extension** of $\mathcal{A}$.

For $S \subseteq A$, let $\mathcal{A}_S$ be the expansion of $\mathcal{A}$ by a constant for every $a \in S$.
Then $\mathcal{A} \prec \mathcal{B}$ iff $\mathcal{A}$ is a substructure of $\mathcal{B}$ and $\mathrm{Th}(\mathcal{A}_A) = \mathrm{Th}(\mathcal{B}_A)$.

Example, continued

$\mathcal{A}_n \not\prec \mathcal{A}_{n+1}$ because $\mathcal{A}_{n+1} \models \exists x \;\; x < -n$ but $\mathcal{A}_n \not\models \exists x \;\; x < -n$

Elementary chains

A chain $(\mathcal{A}_i)_{i\in I}$ is an **elementary chain** if $\mathcal{A}_i \prec \mathcal{A}_j$ for all $i < j$.

Tarski's Chain Lemma

Let $(\mathcal{A}_i)_{i\in I}$ be an elementary chain. Then $\mathcal{A}_i \prec \lim_{i\in I} \mathcal{A}_i$ for all $i \in I$.

Proof.

Let $\mathcal{B} := \lim_{i\in I} \mathcal{A}_i$. Show for any $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n)$, $i \in I$, $\bar{a} \in A_i^n$,

$$\mathcal{B} \models \phi(\bar{a}) \quad \text{iff} \quad \mathcal{A}_i \models \phi(\bar{a})$$

by induction on ϕ.

Base cases

- For atomic formulas $s = t$, use induction on terms to show $t^{\mathcal{B}}(\bar{a}) = t^{\mathcal{A}_i}(\bar{a})$.

- For $R \in \mathcal{L}$

$$\mathcal{B} \models R(\bar{a}) \text{ iff } \bar{a} \in R^{\mathcal{B}} \text{ iff } \bar{a} \in R^{\mathcal{A}_i} \text{ iff } \mathcal{A}_i \models R(\bar{a}).$$

Induction steps

- Straightforward for formulas $\neg\psi$ and $\phi \wedge \psi$.
- For ϕ of the form $\exists y\ \psi(\bar{x}, y)$

$$\begin{aligned}
&\mathcal{B} \models \exists y\ \psi(\bar{a}, y) \\
\text{iff } &\mathcal{B} \models \psi(\bar{a}, b) \text{ for some } b \in B \\
\text{iff } &\mathcal{A}_j \models \psi(\bar{a}, b) \text{ for some } b \in A_j \text{ for some } j > i \text{ by IA} \\
\text{iff } &\mathcal{A}_j \models \exists y\ \psi(\bar{a}, y) \text{ for some } j > i \\
\text{iff } &\mathcal{A}_i \models \exists y\ \psi(\bar{a}, y) \text{ since } \mathcal{A}_i \prec \mathcal{A}_j.
\end{aligned}$$

□

Tarski-Vaught test for elementary substructures

An $\mathcal{L}$-formula $\phi(x)$ is **satisfiable** in an $\mathcal{L}$-structure $\mathcal{A}$ if there exists $a \in A$ such that $\mathcal{A} \models \phi(a)$. Then we say a satisfies $\phi(x)$.

Lemma (Tarski-Vaught test)

Let $\mathcal{B}$ be an $\mathcal{L}$-structure. Then $A \subseteq B$ is the universe of an elementary substructure of $\mathcal{B}$ iff every $\mathcal{L} \cup A$-formula $\phi(x)$ which is satisfiable in $\mathcal{B}_A$ can be satisfied by an element of A.

Proof.

$\Rightarrow$: If $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B}_A \models \exists x\ \phi(x)$, then $\mathcal{A} \models \exists x\ \phi(x)$, i.e., $\mathcal{B} \models \phi(a)$ for some $a \in A$.

$\Leftarrow$: Assume the rhs holds.
Claim 1: A is universe of a substructure $\mathcal{A}$.
For $f \in \mathcal{L}$ and $a_1, \ldots, a_n \in A$, note that $x = f(a_1, \ldots, a_n)$ is satisfiable in $\mathcal{B}_A$. So there is $b \in A$ such that $\mathcal{B}_A \models b = f(a_1, \ldots, a_n)$. Hence A is closed under $f^{\mathcal{B}}$.

Claim 2: $\mathcal{A}_A \models \phi$ iff $\mathcal{B}_A \models \phi$ for every sentence ϕ.
Induct on ϕ: Base cases, negation and conjunction are clear.
Consider $\phi = \exists x \ \psi(x)$.

$$\mathcal{A}_A \models \phi \ \Rightarrow \ \mathcal{A}_A \models \psi(a) \text{ for some } a \in A \ \Rightarrow \ \mathcal{B}_A \models \phi.$$

Conversely,

$$\begin{aligned} & \mathcal{B}_A \models \phi \\ \Rightarrow\ & \psi(x) \text{ is satisfiable in } \mathcal{B} \\ \Rightarrow\ & \psi(x) \text{ is satisfied by some } a \in A \text{ by assumption} \\ \Rightarrow\ & \mathcal{A}_A \models \phi \end{aligned}$$

$\square$

Small elementary substructures from the Tarski-Vaught test

Corollary

Let S be a subset of an $\mathcal{L}$-structure $\mathcal{B}$.
Then $\mathcal{B}$ has an elementary substructure $\mathcal{A}$ containing S of cardinality at most $\max(|S|, |\mathcal{L}|, \aleph_0)$.

Proof.

Construct A as union of an ascending sequence

$$S = S_0 \subseteq S_1 \subseteq \dots$$

Assume S_i is already defined. For every $\mathcal{L} \cup \{S_i\}$-formula $\phi(x)$ that is satisfiable in $\mathcal{B}$, choose $a_\phi \in B$ that satisfies $\phi(x)$.
Let $S_{i+1} := S_i \cup \{a_\phi \mid \phi \text{ is satisfiable}\}$.
Then $A := \bigcup S_i$ is the universe of an elementary substructure $\mathcal{A}$ by the Tarski-Vaught test.
Since there are $\kappa := \max(|S|, |\mathcal{L}|, \aleph_0)$ many $\mathcal{L} \cup \{S\}$-formulas, $|S_1| \leq \kappa$. By induction $|S_i| \leq \kappa$ and so $|A| \leq \kappa$. □