9. Ultraproducts

One more proof of the Compactness Theorem

Goal.
Build a model $\mathcal{M}$ for a finitely satisfiable theory T from the models $\mathcal{M}_\Delta$ for finite $\Delta \subseteq T$.

Filters

A **filter** $\mathcal{F}$ on a set X is a subset of the power set $P(X)$ such that

1. $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$;
2. if $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$; (closed under supersets)
3. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. (closed under intersections)

Example

For every non-empty $A \subseteq X$, the set of supersets of A

$$\mathcal{F}_A := \{B \subseteq X \mid A \subseteq B\}$$

is a filter (the **principal** filter generated by A).

A filter $\mathcal{F}$ is principal iff $\mathcal{F}$ contains a smallest element, i.e. $\bigcap \mathcal{F} \in \mathcal{F}$.

Example

For X infinite, the set of cofinite subsets

$$\mathcal{F} := \{A \subseteq X \mid X \setminus A \text{ is finite}\}$$

is a filter (the **Fréchet filter** on X).

Maximal filters

A filter $\mathcal{F}$ on a set X is an **ultrafilter** if for every $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Facts.

1. $\mathcal{F}$ is an ultrafilter iff $\mathcal{F}$ is a maximal filter under inclusion.
2. (**Ultrafilter Lemma**) Every filter is contained in an ultrafilter. [Follows from Zorn's Lemma.]

Example

1. An ultrafilter is principal iff it contains a singleton.
2. The Fréchet filter is not contained in any principal ultrafilter.

Intuition: Elements of an ultrafilter are the "big" subsets of X; the others are "small".

Ultraproducts

Let $\mathcal{M}_x$ be an $\mathcal{L}$-structure for any $x \in X$, let $\mathcal{F}$ be an ultrafilter on X.

1. On $\prod_{x \in X} M_x := \{f \colon X \to \bigcup M_x \mid f(x) \in M_x \text{ for all } x \in X\}$ define an equivalence relation $\sim$ by
$$f \sim g \text{ if } \{x \in X \mid f(x) = g(x)\} \in \mathcal{F}.$$
2. On the set of equivalence classes $[g]$ define an $\mathcal{L}$-structure $\mathcal{M} = \prod_{x \in X} \mathcal{M}_x / \mathcal{F}$ (ultraproduct of the $\mathcal{M}_x$) as follows:
3. For a constant $c \in \mathcal{L}$,
$$c^{\mathcal{M}} := [x \mapsto c^{\mathcal{M}_x}].$$
4. For a relation $R \in \mathcal{L}$ of arity n,
$$R^{\mathcal{M}}([g_1], \dots, [g_n]) \text{ if } \{x \in X \mid R^{\mathcal{M}_x}(g_1(x), \dots, g_n(x))\} \in \mathcal{F}.$$

5. For a function $f \in \mathcal{L}$ of arity n,

$$f^{\mathcal{M}}([g_1], \ldots, [g_n]) = [h] \text{ if } \{x \in X | f^{\mathcal{M}_x}(g_1(x), \ldots, g_n(x)) = h(x)\} \in \mathcal{F}.$$

Note.

1. $f^{\mathcal{M}}$ reduces to 4. by viewing the graph of a function as relation.
2. The interpretations above are well-defined (HW).
3. If $\mathcal{F}$ is just a filter, the construction still works.

Intuition: The ultraproduct $\prod_{x \in X} \mathcal{M}_x / \mathcal{F}$ "averages" over the $\mathcal{M}_x$.

Łoś's Theorem

Theorem (Łoś)

For any $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n)$,

$$\prod_{x \in X} \mathcal{M}_x \,/\mathcal{F} \models \phi([g_1], \ldots, [g_n])$$
$$\text{iff } \{x \in X \mid \mathcal{M}_x \models \phi(g_1(x), \ldots, g_n(x))\} \in \mathcal{F}.$$

Proof.

By induction on ϕ. Let $\mathcal{M} := \prod_{x \in X} \mathcal{M}_x \,/\mathcal{F}$.
Consider the induction step for ϕ of the form $\exists y \; \psi(\bar{x}, y)$:

$$\begin{aligned}
&\mathcal{M} \models \exists y \; \psi(\overline{[g]}, y) \\
\text{iff } &\mathcal{M} \models \psi(\overline{[g]}, [f]) \text{ for some } [f] \in M \\
\text{iff } &\{x \in X \mid \mathcal{M}_x \models \psi(\overline{g(x)}, f(x))\} \in \mathcal{F} \text{ for some } f \in \prod M_x \\
\text{iff } &\{x \in X \mid \mathcal{M}_x \models \exists y \; \psi(\overline{g(x)}, y)\} \in \mathcal{F}.
\end{aligned}$$

For the last equivalence note that for every $f \in \prod M_x$,

$$\{x \in X \mid \mathcal{M}_x \models \psi(\overline{g(x)}, f(x))\} \subseteq \{x \in X \mid \mathcal{M}_x \models \exists y\ \psi(\overline{g(x)}, y)\}.$$

For $\supseteq$, pick for each $x \in X$ with $\mathcal{M}_x \models \exists y\ \psi(\overline{g(x)}, y)$ a witness $c_x \in M_x$ such that $\mathcal{M}_x \models \psi(\overline{g(x)}, c_x)$.
Define

$$f\colon X \to \bigcup_{x \in X} M_x,\ x \mapsto \begin{cases} c_x & \text{if } \mathcal{M}_x \models \exists y\ \psi(\overline{g(x)}, y), \\ \text{arbitrary} & \text{else.} \end{cases}$$

Equality of the above index sets follows as well as the last equivalence on the previous slide. $\square$

Ultrapowers of $\mathcal{M}$ are elementary equivalent to $\mathcal{M}$

An **ultrapower** $\mathcal{M}^X/\mathcal{F}$ is an ultraproduct $\prod_{x\in X}\mathcal{M}_x/\mathcal{F}$ with $\mathcal{M}_x = \mathcal{M}$ for all $x \in X$.

Corollary

$\mathrm{Th}(\mathcal{M}^X/\mathcal{F}) = \mathrm{Th}(\mathcal{M})$.

Proof.

Immediate from Łoś's Theorem. □

Existence of non-standard models, 2

Example

Let $\mathbb{N} := (\mathbb{N}, +, \cdot, 0, 1, <)$ with the usual interpretation of symbols.
Let $\mathcal{F}$ be a non-principal ultrafilter on ω (containing the Fréchet filter).
Then there exists $a := [(0, 1, 2, \dots)]$ in the ultrapower $\mathbb{N}^\omega / \mathcal{F}$ such that for every $n \geq 1$

$$\mathbb{N}^\omega / \mathcal{F} \models \underbrace{1 + \cdots + 1}_{n \text{ times}} < a.$$

Proof.

1. For fixed n, we have $\mathbb{N} \models n < a(x)$ for all but finitely many $x < \omega$.
2. Since every cofinite set is an element of $\mathcal{F}$, the claim follows by Łoś's Theorem. □

The hyperreals $*\mathbb{R}$

Let $\mathbb{R} := (\mathbb{R}, +, \cdot, 0, 1, <)$ with the usual interpretation of symbols denote the ordered field of the reals.
For a non-principal ultrafilter $\mathcal{F}$ on ω,

$$*\mathbb{R} := \mathbb{R}/\mathcal{F}$$

is the **field of hyperreals**.

- $\mathrm{Th}(*\mathbb{R}) = \mathrm{Th}(\mathbb{R})$.
- $|*\mathbb{R}| = 2^{\aleph_0}$.
- $*\mathbb{R}$ has **infinitesimal** elements a such that for all $n \in \mathbb{N}$

  $$*\mathbb{R} \models 0 < a \wedge (\underbrace{1 + \cdots + 1}_{n}) \cdot a < 1$$

  e.g. $a = [(1, \frac{1}{2}, \frac{1}{3}, \dots)]$.
- Abraham Robinson used this to develop **non-standard analysis** rigorously formalizing the use of infinitesimals and infinite numbers in calculus as proposed by Leibniz.

Compactness via ultraproducts

Compactness Theorem

Every finitely satisfiable theory T is satisfiable.

Proof.

1. Let $X := P_{\mathrm{fin}}(T)$ be the set of finite subsets of T.
2. Assume every $\Delta \in X$ has a model $\mathcal{M}_\Delta$.
3. For $\phi \in T$, let $X_\phi := \{\Delta \in X \mid \mathcal{M}_\Delta \models \phi\}$.
4. Let $\mathcal{F}$ be the set of all intersections of X_ϕ for finitely many $\phi \in T$ and their supersets.
5. Since any finite intersection $X_{\phi_1} \cap \cdots \cap X_{\phi_n}$ contains $\{\phi_1, \ldots, \phi_n\}$, it is non-empty. Hence $\mathcal{F}$ is a filter.
6. By the Ultrafilter Lemma, $\mathcal{F}$ is contained in some ultrafilter $\mathcal{U}$.
7. For $\phi \in T$, we have $X_\phi \in \mathcal{U}$ and $\prod_{\Delta \in X} M_\Delta/\mathcal{U} \models \phi$ by Łoś's Theorem.
8. Thus $\prod_{\Delta \in X} M_\Delta/\mathcal{U} \models T$. □