8. Applications of Compactness

Existence of non-standard models

Let $\mathbb{N} := (\mathbb{N}, +, \cdot, 0, 1, <)$ with the usual interpretation of symbols.
$\mathrm{Th}\,\mathbb{N}$ is called **true arithmetic**.
Its model $\mathbb{N}$ is called the **standard model** of arithmetic.

Theorem

There exist a structure $\mathcal{M}$ such that $\mathcal{M} \models \mathrm{Th}(\mathbb{N})$ and $a \in M$ such that $a > \underbrace{1 + \cdots + 1}_{n \text{ times}}$ for all $n \in \mathbb{N}$.

Proof.

1. Expand the language by a new constant symbol c.
2. $T := \mathrm{Th}(\mathbb{N}) \cup \{1 < c, 1 + 1 < c, 1 + 1 + 1 < c, \dots\}$ is finitely satisfiable.
3. By the Compactness Theorem, T has a model $\mathcal{M}'$ with $a := c^{\mathcal{M}'}$ as required.
4. Removing c from the expanded language again, we obtain a **reduct** $\mathcal{M}$ of $\mathcal{M}'$ such that $\mathcal{M} \equiv \mathbb{N}$ (HW). □

Showing that a class is not elementary

A group $(G, \cdot, ^{-1}, 1)$ is **torsion** if every $x \in G$ has finite order.

Theorem

The class of torsion groups $\mathcal{T}$ is not elementary.

Proof.

1. For $n \geq 1$, let $\phi_n(x)$ be $x^1 \neq 1 \wedge \cdots \wedge x^n \neq 1$.
2. Expand the language by a new constant symbol c. Then

$$\mathrm{Th}(\mathcal{T}) \cup \{\phi_1(c), \phi_2(c), \phi_3(c), \ldots\}$$

   is finitely satisfiable, hence has a model $\mathcal{M}$ in which $c^{\mathcal{M}}$ has infinite order.
3. Thus $\mathrm{Th}(\mathcal{T})$ does not axiomatize the class of torsion groups. □