7. Henkin construction

Goal

Every finitely satisfiable theory can be extended to a **maximal**, finitely satisfiable theory T **with witness property**,
[i.e., for every $\mathcal{L}$-formula $\phi(x)$ there exists a constant $c \in \mathcal{L}$ s.t.

$$T \models (\exists x\, \phi(x)) \rightarrow \phi(c). \qquad]$$

Lemma (Marker 2.1.8)

Let T be a finitely satisfiable $\mathcal{L}$-theory. There exist

- $\mathcal{L}^* \supseteq \mathcal{L}$ (with $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$) and
- a finitely satisfiable $\mathcal{L}^*$-theory $T^* \supseteq T$ with witness property.

Proof.

Define

- $\mathcal{L}_1 := \mathcal{L} \cup \{c_\phi \mid \phi(x) \text{ an } \mathcal{L}\text{-formula}\}$, with c_ϕ a **new** constant,
- $T_1 := T \cup \{\underbrace{(\exists x\ \phi(x)) \to \phi(c_\phi)}_{=:\Theta_\phi} \mid \phi(x) \text{ and } \mathcal{L}\text{-formula}\}$.

Claim. T_1 is finitely satisfiable.

1. Let $\Delta \subseteq T_1$ finite, say $\Delta = (\Delta \cap T) \cup \{\Theta_{\phi_1}, \dots, \Theta_{\phi_n}\}$.
2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure such that $\mathcal{M} \models \Delta \cap T$.
3. Expand $\mathcal{M}$ to an $\mathcal{L}_1$-structure $\mathcal{M}_1$ by adding interpretations of the new constants,

$$c_\phi^{\mathcal{M}_1} := \begin{cases} a & \text{if } \mathcal{M} \models \exists x\ \phi(x) \text{ and } \mathcal{M} \models \phi(a) \text{ for } a \in M, \\ b & \text{else, for any } b \in M. \end{cases}$$

4. Then $\mathcal{M}_1 \models \Delta$.

Note: There may now be $\mathcal{L}_1$-formulas $\exists x\ \phi(x)$ without witness. So iterate the construction to get

- $\mathcal{L} =: \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \dots$
- $T =: T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$

such that for any $\mathcal{L}_i$-formula ϕ there exists $c_\phi \in \mathcal{L}_{i+1}$ such that

$$T_{i+1} \models (\exists x\ \phi(x)) \rightarrow \phi(\bar{c}_\phi).$$

1. Then $T^* := \bigcup_{i \in \mathbb{N}} T_i$ has the witness property over $\mathcal{L}^* := \bigcup_{i \in \mathbb{N}} \mathcal{L}_i$ and is finitely satisfiable since every T_i is.
2. Since there are $\leq |\mathcal{L}_i| + \aleph_0$ formulas in $\mathcal{L}_i$, it follows that $\mathcal{L}^*$ has cardinality $\leq |\mathcal{L}| + \aleph_0$. □

Lemma (Marker 2.1.9)

Let T be a finitely satisfiable $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. Then either $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable.

Corollary (Marker 2.1.10)

Every finitely satisfiable theory T is contained in a maximal, finitely satisfiable theory.

Proof.

By Zorn's Lemma, T is contained in a fin satisfiable theory T^* that is maximal under inclusion.
By Lemma 2.1.9, T^* is maximal as a theory. □

The Compactness Theorem (strengthened by cardinality)

Theorem

Let T be a finitely satisfiable $\mathcal{L}$-theory and $|\mathcal{L}| \leq \kappa$ infinite.
Then T has a model of cardinality $\leq \kappa$.

Proof.

1. By Lemma 2.1.8, we have $\mathcal{L}^* \supseteq \mathcal{L}$ (with $|\mathcal{L}^*| \leq \kappa$) and a finitely satisfiable $\mathcal{L}^*$-theory $T^* \supseteq T$ with witness property.
2. By Cor 2.1.10, we have a maximal, finitely satisfiable $T' \supseteq T^*$ with witness property.
3. By Lemma 2.1.7, T' has a model $\mathcal{M}'$ (an $\mathcal{L}^*$-structure).
4. Removing symbols from $\mathcal{L}^* \setminus \mathcal{L}$ from $\mathcal{M}'$ to get an $\mathcal{L}$-**reduct** $\mathcal{M}$ satisfying T. □

Review

1. **Henkin construction**: a model whose universe is the set of constant symbols of the language.
2. By replacing “finitely satisfiable” by “consistent”, Gödel’s Completeness Theorem can be shown similar to the Compactness Theorem above.