6. Compactness Theorem

Henkin constructions

A theory T is **finitely satisfiable** if every finite subset of T is satisfiable.

Compactness Theorem

Every finitely satisfiable theory is satisfiable.

Henkin's proof idea: Build a model $\mathcal{M}$ for T over a language expanded by enough constants to name every element in $\mathcal{M}$.

An $\mathcal{L}$-theory T has the **witness property** if for every $\mathcal{L}$-formula $\phi(y)$ with one free variable y there exists a constant $c \in \mathcal{L}$ s.t.

$$T \models (\exists y\, \phi(y)) \to \phi(c).$$

A theory T is **maximal** if for every ϕ either $\phi \in T$ or $\neg\phi \in T$.

Lemma (Marker 2.1.6)

Let T be maximal and finitely satisfiable, let $\Delta \subseteq T$ be finite.
If $\Delta \models \psi$, then $\psi \in T$.

Proof by contradiction.

Suppose $\psi \notin T$. Then $\neg\psi \in T$ by maximality and $\Delta \cup \{\neg\psi\}$ is a finite unsatisfiable subset of T. Contradiction. $\square$

Maximal, fin satisfiable T with witness property has model

Lemma (Marker 2.1.7)

Let T be a maximal, finitely satisfiable $\mathcal{L}$-theory with witness property. Then T has a model $\mathcal{M}$.

Proof.

Let $\mathcal{C}$ be the set of constant symbols in $\mathcal{L}$.
For $c, d \in \mathcal{C}$, define $c \sim d$ if $T \models c = d$.
Claim 1. $\sim$ is an equivalence on $\mathcal{C}$ by Lemma 2.1.6.

- **Universe of the model $\mathcal{M}$.**
  $M := \mathcal{C}/\sim$, the set of equivalence classes c^* for $c \in \mathcal{C}$.
- **Interpretation of constant symbols.**

$$c^{\mathcal{M}} := c^*$$

- **Interpretation of n-ary relation symbols R.**

$$R^{\mathcal{M}} := \{(c_1^*, \ldots, c_n^*) \mid R(c_1, \ldots, c_n) \in T\}$$

Well-defined since $R(\bar{c}) \in T$ iff $R(\bar{d}) \in T$ for all $c_1 \sim d_1, \ldots, c_n \sim d_n$ by Lemma 2.1.6.

- **Interpretation of n-ary function symbols f.**

$$f^{\mathcal{M}}(c_1^*, \ldots, c_n^*) := d^* \ \text{ if } \ f(c_1, \ldots, c_n) = d \ \in T$$

Well-defined since for all $\bar{c} = (c_1, \ldots, c_n) \in \mathcal{C}^n$:

1. (Image exists)
   $\emptyset \models \exists y \ f(\bar{c}) = y$ and the witness property for T yields that there is $d \in \mathcal{C}$ such that $f(\bar{c}) = d \in T$.
2. (Image is unique)
   If $f(\bar{c}) = a \in T$ and $f(\bar{d}) = b \in T$ for $c_1 \sim d_1, \ldots, c_n \sim d_n$, then $f(\bar{c}) = f(\bar{d}) \in T$ and $a \sim b$ by Lemma 2.1.6.

Note. $\mathcal{M}$ is uniquely determined by $\mathcal{L}$ and the atomic formulas in T.

The interpretation of terms is well-behaved

Claim 3. For any $\mathcal{L}$-term $t(x_1, \ldots, x_n)$ and $c_1, \ldots, c_n, d \in \mathcal{C}$

$$t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^* \text{ iff } t(c_1, \ldots, c_n) = d \in T.$$

$\Leftarrow$ follows by induction on t.
$\Rightarrow$: Assume $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$.
By the witness property, we have $e \in \mathcal{C}$ such that $t(c_1, \ldots, c_n) = e \in T$.
Then $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = e^*$ by $\Leftarrow$.
So $d^* = e^*$, $d = e \in T$ and $t(c_1, \ldots, c_n) = d \in T$.

Show $\mathcal{M} \models T$

Claim 4. For any $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n)$ and $c_1, \ldots, c_n \in \mathcal{C}$

$$\mathcal{M} \models \phi(\bar{c}^*) \text{ iff } \phi(\bar{c}) \in T.$$

Induction on Φ:
Base cases:

- Assume ϕ is $t_1 = t_2$. By the witness property we have $d_1, d_2 \in \mathcal{C}$ such that $t_1(\bar{c}) = d_1, t_2(\bar{c}) = d_2$ are in T.
  Then $t_i^{\mathcal{M}}(\bar{c}^*) = d_i^*$ for $i = 1, 2$ by Claim 3 and

  $$\mathcal{M} \models \phi(\bar{c}^*) \quad \text{iff} \quad d_1^* = d_2^* \quad \text{iff} \quad d_1 = d_2 \in T \quad \text{iff} \quad t_1(\bar{c}) = t_2(\bar{c}) \in T$$

- Similar for $R(t_1, \ldots, t_m)$.

Induction steps:

- ...
- ...
- If ϕ is $\exists y\ \psi(\bar{x}, y)$, then

$$\begin{aligned}\mathcal{M} \models \phi(\bar{c}^*) &\text{ iff } \mathcal{M} \models \psi(\bar{c}^*, d^*) \text{ for some } d \in \mathcal{C} \\ &\text{ iff } \psi(\bar{c}, d) \in T \text{ for some } d \in \mathcal{C} \text{ by induction assumption} \\ &\text{ iff } \exists y\ \psi(\bar{c}, y) \in T \text{ by witness property for } \Leftarrow\end{aligned}$$

Lemma 2.1.7 is proved. $\square$

Review

1. Where was the **maximality** of T used?
2. Where was the **witness property** for T used?

Outlook

Every finitely satisfiable theory can be extended to a maximal, finitely satisfiable theory with witness property.