5. Gödel's Completeness Theorem

Entailment and proofs

Let T be an $\mathcal{L}$-theory, ϕ an $\mathcal{L}$-sentence.

- How to show $T \not\models \phi$?
  Find some $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models T$ but $\mathcal{M} \not\models \phi$.
- How to show $T \models \phi$?
  - Check all models of T?
  - Better: Prove ϕ from T.

A **proof** ϕ from T is a finite sequence of $\mathcal{L}$-formulas $\psi_1, \ldots, \psi_n$ such that

- $\psi_n = \phi$ and
- $\psi_i \in T$ or ψ_i follows from $\psi_1, \ldots, \psi_{i-1}$ by "simple logical rules" for every $i \leq n$.

Then write $T \vdash \phi$ (read T **proves** ϕ).

Sketch of a proof system

There are many choices for particular "simple logical rules", e.g:

- Propositional rules: From ϕ and ψ conclude $\phi \wedge \psi$.
- Equality rules: From $\bigwedge_{i=1}^{n} s_i = t_i$ and $R(s_1, \ldots, s_n)$ conclude $R(t_1, \ldots, t_n)$.
- Quantifier rules: From $\phi(t, x)$ conclude $\exists y\ \phi(y, x)$.

Note: Proof systems are defined for f.o. formulas (not sentences) and quantifiers require some care. See

- Enderton. A Mathematical Introduction to Logic. 2nd ed., Harcourt, 2002.
- Ebbinghaus, Flum, Thomas. Mathematical Logic. 3rd ed., Springer, 2021.

Properties of proof systems

- Proofs are finite.
- (**Soundness**) If $T \vdash \phi$, then $T \models \phi$.
- For finite T, there exists an algorithm that on input ϕ and $\psi_1, \ldots, \psi_n$, decides whether $\psi_1, \ldots, \psi_n$ is a proof of ϕ from T.

Theorem
For a computable theory T over a computable language $\mathcal{L}$,

$$\{\phi \mid T \vdash \phi\}$$

is recursively enumerable.

Proof.
Sketch algorithm that accepts ϕ if $T \vdash \phi$; does not halt if $T \nvdash \phi$.

1. Let $\sigma_0, \sigma_1, \ldots$ be a computable enumeration of all finite sequences of $\mathcal{L}$-formulas [Exists since $\mathcal{L}$ is computable].
2. At stage i, algorithm checks whether σ_i is a proof of ϕ from T. [Check whether each ψ in σ_i is either in T (computable) or follows from previous formulas by logical rules.]
3. Algorithm answers "yes" if σ_i is proof of ϕ; else goes to stage $i + 1$.

□

Example

Since the axioms of ZFC are computable, $\{\phi \mid \text{ZFC} \vdash \phi\}$ is recursively enumerable (but not computable if ZFC is consistent by Gödel's Second Incompleteness Theorem).

Gödel's Completeness Theorem

For any $\mathcal{L}$-theory T and any $\mathcal{L}$-sentence ϕ,

$$T \models \phi \text{ iff } T \vdash \phi.$$

[$\Leftarrow$ is soundness of the proof system; $\Rightarrow$ is its **completeness**.]

Proof.

See Enderton or Ebbinghaus et al. □

T is **consistent** if $T \nvdash \bot$.

Corollary (Henkin)

T is consistent iff T is satisfiable.

Proof.

$\Leftarrow$ is clear.

$\Rightarrow$ follows by contraposition from the Completeness Thm. □

Compactness as a consequence of completeness

How to get models for infinite theories?

Compactness Theorem

T is satisfiable iff every finite subset of T is satisfiable.

Proof.

$\Rightarrow$: clear

$\Leftarrow$: Assume T is **not** satisfiable.

1. Then T is inconsistent by the Completeness Theorem.
2. Let σ be a proof for $\perp$ from T.
3. The set Δ of $\psi \in T$ that occur in σ is finite and $\Delta \vdash \perp$.
4. Hence Δ is not satisfiable. $\square$

Why the name 'Compactness Theorem'?

The Compactness Theorem states that the topological space of complete $\mathcal{L}$-theories

$$\mathcal{T} := \{\mathrm{Th}(\mathcal{A}) \mid \mathcal{A} \text{ is an } \mathcal{L}\text{-structure}\}$$

is compact.

Basic open sets are $U_\phi := \{T \in \mathcal{T} \mid \phi \in T\}$ for a sentence ϕ.

1. Consider an open cover $\bigcup_{\phi \in S} U_\phi = \mathcal{T}$ for a set S of sentences, i.e. every $T \in \mathcal{T}$ contains some $\phi \in S$.
2. Suppose there is no finite subcover, i.e. for every finite $\Delta \subseteq S$ we have $T \in \mathcal{T}$ such that $\Delta \cap T = \emptyset$, hence $\{\neg\phi \mid \phi \in \Delta\} \subseteq T$.
3. Then $\{\neg\phi \mid \phi \in S\}$ is satisfiable by the Compactness Theorem, hence contained in some $T \in \mathcal{T}$, which cannot be contained in any U_ϕ for $\phi \in S$. Contradiction.
4. Thus every open cover of $\mathcal{T}$ has a finite subcover.