4. Elementary equivalence

Elementary equivalence

$\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ are **elementary equivalent** (short $\mathcal{A} \equiv \mathcal{B}$) if they satisfy the same sentences, i.e.,

$$\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B}).$$

Lemma

If $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$.

The converse holds only for finite structures (HW).

Proof.

Let $h \colon \mathcal{A} \to \mathcal{B}$ be an $\mathcal{L}$-isomorphism. We show that

$$\mathcal{A} \models \phi(a_1, \ldots, a_n) \text{ iff } \mathcal{B} \models \phi(h(a_1), \ldots, h(a_n))$$

for all $\mathcal{L}$-formulas ϕ by induction.

Let $\bar{a} := (a_1, \ldots, a_n) \in A^n$ and write $h(\bar{a}) := (h(a_1), \ldots, h(a_n))$.
Claim 1: For every term $t(x_1, \ldots, x_n)$

$$h(t^{\mathcal{A}}(\bar{a})) = t^{\mathcal{B}}(h(\bar{a})).$$

Induct on t.
Base cases:

- If $t = x_i$ a variable, then $h(t^{\mathcal{A}}(\bar{a})) = h(a_i) = t^{\mathcal{B}}(h(\bar{a}))$.
- If $t = c$ a constant, then ...

Induction step:
If $t = f(t_1, \ldots, t_m)$, then

$$\begin{aligned} h(t^{\mathcal{A}}(\bar{a})) &= h(f^{\mathcal{A}}(t^{\mathcal{A}}(\bar{a}), \ldots, t_m^{\mathcal{A}}(\bar{a}))) \\ &= f^{\mathcal{B}}(h(t^{\mathcal{A}}(\bar{a})), \ldots, h(t_m^{\mathcal{A}}(\bar{a}))) \\ &= f^{\mathcal{B}}(t^{\mathcal{B}}(h(\bar{a}))), \ldots, t_m^{\mathcal{B}}(h(\bar{a})))) \\ &= t^{\mathcal{B}}(h(\bar{a})). \end{aligned}$$

Claim 2: For every formula $\phi(x_1, \ldots, x_n)$

$$\mathcal{A} \models \phi(\bar{a}) \text{ iff } \mathcal{B} \models \phi(h(\bar{a})).$$

Induct on ϕ.
Base cases:

- If ϕ is $t_1 = t_2$, then

$$\begin{aligned}
\mathcal{A} \models \phi(\bar{a})) &\text{ iff } t_1^{\mathcal{A}}(\bar{a}) = t_2^{\mathcal{A}}(\bar{a}) \\
&\text{ iff } h(t_1^{\mathcal{A}}(\bar{a})) = h(t_2^{\mathcal{A}}(\bar{a})) \text{ since } h \text{ injective} \\
&\text{ iff } t_1^{\mathcal{B}}(h(\bar{a})) = t_2^{\mathcal{B}}(h(\bar{a})) \text{ by Claim 1} \\
&\text{ iff } \mathcal{B} \models \phi(h(\bar{a}))
\end{aligned}$$

- ...

Induction steps:

- ...
- ...
- If ϕ is $\exists y\ \psi(\bar{x}, y)$, then

$$\begin{aligned}\mathcal{A} \models \phi(\bar{a})) &\text{ iff } \mathcal{A} \models \psi(\bar{a}, c) \text{ for some } c \in A \\ &\text{ iff } \mathcal{B} \models \psi(h(\bar{a}), b) \text{ for some } b \in A \text{ since } b \text{ is surjective} \\ &\text{ iff } \mathcal{B} \models \phi(h(\bar{a})).\end{aligned}$$

Definable sets are invariant under automorphisms

Lemma (cf. Marker, Prop 1.3.5)

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. If $R \subseteq M^n$ is definable in $\mathcal{M}$, then $h(R) = R$ for all automorphisms h of $\mathcal{M}$.

Proof.

Let $\phi(\bar{x})$ such that

$$R = \{\bar{a} \in M^n \mid \mathcal{M} \models \phi(\bar{a})\}.$$

By the previous proof, for any automorphism h of $\mathcal{M}$,

$$\mathcal{M} \models \phi(\bar{a}) \quad \text{iff} \quad \underbrace{h(\mathcal{M})}_{=\mathcal{M}} \models \phi(h(\bar{a}))$$

Hence $h(R) = R$. $\square$

Application: $\mathbb{R}$ is not definable in the field of the complex numbers

Corollary (Marker, Cor 1.3.6)

$\mathbb{R}$ is not definable in $(\mathbb{C}, +, -, 0, \cdot, 1)$.

Proof.

Let $r \in \mathbb{R}$ and $s \in \mathbb{C} \setminus \mathbb{R}$ be algebraically independent over $\mathbb{Q}$, i.e., there exists no $p(x, y) \in \mathbb{Q}[x, y]$ such that $p(r, s) = 0$.

Fact (using AC). There exists an automorphism h of $\mathbb{C}$ such that $h(r) = s$.

Hence $\mathbb{R}$ cannot be definable. □

Equivalent theories

$\mathcal{L}$-theories S, T are **equivalent** (short $S \equiv T$) if $S \models T$ and $T \models S$ (i.e., S and T have the same classes of models).

Theorem

For a satisfiable theory T TFAE:

1. T is complete.
2. All models of T are elementary equivalent.
3. There exists a structure $\mathcal{A}$ such that $T \equiv \mathrm{Th}(\mathcal{A})$.

Proof.

1.⇒3. Let $\mathcal{A}$ be a model for T. If $\mathcal{A} \models \phi$, then $T \not\models \neg\phi$.
So $T \models \phi$ by completeness. Thus $T \models \mathrm{Th}(\mathcal{A})$; the converse is clear.
3.⇒2. Let $\mathcal{B} \models \mathrm{Th}(\mathcal{A})$. Then $\mathrm{Th}(\mathcal{A}) \subseteq \mathrm{Th}(\mathcal{B})$.
Conversely, if $\phi \in \mathrm{Th}(\mathcal{B})$, then $\mathcal{A}$ cannot satisfy $\neg\phi$. So $\phi \in \mathrm{Th}(\mathcal{A})$.
2.⇒1. HW □