3. First order theories

First order theories

1. An $\mathcal{L}$-**theory** is a set T of $\mathcal{L}$-sentences.
2. An $\mathcal{L}$-structure $\mathcal{A}$ is a **model of** a theory T if $\mathcal{A} \models \phi$ for all $\phi \in T$ (write $\mathcal{A} \models T$, read $\mathcal{A}$ **satisfies** T).
3. A theory T **entails** a sentence ϕ (write $T \models \phi$) if every model of T satisfies ϕ.
4. A theory T is **satisfiable** if T has a model.
   Note T is satisfiable iff $T \not\models \bot$.
   - Assume T has a model $\mathcal{A}$. Then $\mathcal{A} \not\models \bot$. So $T \not\models \bot$.
   - Conversely, assume T has no models. Then $T \models \bot$ trivially.
5. A theory T is **complete** if for every sentence ϕ either $T \models \phi$ or $T \models \neg\phi$.
6. The **(complete) theory of a structure** $\mathcal{A}$ is

$$\mathrm{Th}(\mathcal{A}) = \{\phi \text{ sentence} \mid \mathcal{A} \models \phi\}.$$

   - $\mathrm{Th}(\mathcal{A})$ is complete since for every sentence ϕ either $\mathcal{A} \models \phi$ or not.

Elementary classes

- A class of $\mathcal{L}$-structures $\mathcal{K}$ is an **elementary class** if there is an $\mathcal{L}$-theory T such that

$$\mathcal{K} = \{\mathcal{M} \mid \mathcal{M} \models T\}.$$

- Then T is a set of **axioms** for $\mathcal{K}$.

Example: Groups

Recall $\mathcal{L}_{\text{Groups}} = \{\cdot, ^{-1}, 1\}$. Let T be the set containing

- $\forall x \forall y \forall z\ x(yz) = (xy)z$
- $\forall x\ 1x = x$
- $\forall x\ x^{-1}x = 1$

Then $\mathcal{A} \models T$ iff $\mathcal{A}$ is a group.

- T is satisfiable (every group is a model).
- $T \models \forall x\ x1 = x$ and $T \models \forall x\ xx^{-1} = 1$.
- T is not complete.
  E.g. for $\phi = \forall x \forall y\ xy = yx$ we have $T \not\models \phi$ and $T \not\models \neg\phi$
  since there are non-abelian and abelian groups

Example: Graphs

Recall $\mathcal{L}_{\mathrm{Graph}} = \{E\}$.
Simple undirected graphs are axiomatized by

- $\forall x \; \neg E(x, x)$
- $\forall x \forall y \; (E(x, y) \to E(y, x))$

Example: Algebraically closed fields (ACF)

A field F is **algebraically closed** if every non-constant polynomial over F has a root in F.

Algebraically closed fields (ACF) are axiomatized by the usual first order field axioms and

- $\forall a_0 \ldots \forall a_n \exists x\ x^{n+1} + \sum_{i=0}^{n} a_i x^i = 0$

for all $n \in \mathbb{N}$.

Example: Zermelo Fraenkel set theory

ZF is an infinite set of first order sentences over the language $\{\in\}$, e.g.

- $\exists x \forall y \; \neg(y \in x)$
- $\forall x \forall y \; (\forall z \; z \in x \leftrightarrow z \in y) \rightarrow x = y$
- $\ldots$

Is ZF satisfiable? Complete?

Example: Computability

Consider $\mathbb{N}$ as structure over $\mathcal{L} = \{+, \cdot, 0, 1\}$.

- Definable subsets of $\mathbb{N}$ are called **arithmetical** in CS.
- A subset S of $\mathbb{N}^k$ is **decidable** (**computable**, **recursive**) if there exists an algorithm that on input $x \in \mathbb{N}^k$ answers "yes" if $x \in S$ and "no" if $x \notin S$.
- There exists an $\mathcal{L}$-formula $T(e, x, s)$ such that $\mathbb{N} \models T(e, x, s)$ iff the **Turing machine with code** e halts on input x in $\leq s$ steps.
- Hence the **Halting Problem**

$$H = \{(e, x) \in \mathbb{N}^2 \mid T(e, x, s) \text{ for some } s \in \mathbb{N}\}$$

  is definable.
- Fact: The Halting Problem is undecidable.

Theorem

The theory of the natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ is undecidable.

- A subset S of $\mathbb{N}^k$ is **recursively enumerable** if there exists an algorithm that enumerates the elements in S.
- Fact: By Matijasevič's solution to **Hilbert's 10th Problem**, $S \subseteq \mathbb{N}^k$ is recursively enumerable iff S is **diophantine**, i.e., there exists a polynomial $p(x_1, \ldots, x_k, y_1, \ldots, y_\ell)$ over $\mathbb{Z}$ such that

$$S = \{(a_1, .., a_k) \in \mathbb{N}^k \mid \mathbb{N} \models \exists y_1 .. \exists y_\ell p(a_1, .., a_k, y_1, .., y_\ell) = 0\}.$$